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Systematic Escape Using Billiard Moves

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***Abstract*— We consider a robot that can move in a straight line, until it hits any wall of a given 2D rectangle room. If the robot hits a corner of the room then it will stop, otherwise the robot bounces off the wall using the laws of reflection and again move in another straight line. The room may contain an unbroken opening of unit length on a wall, which is unknown to the robot. If the robot reaches any point of the opening with a non-zero angle, then it escapes through the opening. The objective of the problem is to devise an algorithm for the robot which enables it to find if there is an opening on the perimeter of any given rectangle or detect that the room contains no opening. This work is a continuation of our previous publication [1], where we presented an algorithm that enables the robot to find the opening or correctly declare that there is none, when any two adjacent sides of the rectangle are integer and co-prime. In this work, we study the problem for rectangles of any lengths and we have proposed a strategy where the robot is guaranteed to find the opening or correctly declare that there is none. Additionally, We provide other interesting results related to our proposed algorithm.**

I. Introduction

This work is an extension of our previous work [1]. For the sake of completeness, we discuss here some of the concepts, which were described in our previous work.

We first consider the concept of a symmetric bounce. If a robot hits an obstacle and changes its direction, the action is called bouncing. The point on the obstacle where the bouncing robot path meets the obstacle is called the "bouncing point". The path of the bouncing robot before the bounce makes an angle with the perpendicular to the tangent of the obstacle. which is called the "incident angle" of the bounce. Similarly, the path of the bouncing robot after the bounce makes an angle with the same perpendicular to the tangent of the obstacle, which is called the "reflection angle" of the bounce. When the incident angle and the reflection angle are equal, the bouncing is called *symmetric bouncing*. This above definition is given in the context of a generic obstacle. If the obstacle is a straight line, then the incident angle and reflection angle are measured from the perpendicular of that straight line. Refer to Figure 1.

We consider a rectangle which is denoted by $(a \times b)$, whose sides are of length $a, b \in \mathbb{R}_+, a \leq b$. There is a possibility that there is an opening on the perimeter of unit length. If the opening is present, then there is only one opening which can not span across two sides. If in practice the opening length is not unit length, then we can always scale the rectangle such a way that the opening becomes unit length. Hence, we can conclude $a, b \geq 1$.

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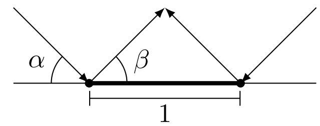

Fig. 1: An example of a symmetric bounce on the wall made by the robot. α is the angle of incidence, β is the angle of reflection. Here, $\alpha = \beta$. The discovered portion of the wall is indicated by the thick line segment. The bouncing points are indicated by a black dot.

Once the robot starts moving it will not stop unless it hits a corner or finds the opening. Thus, once the robot starts its movement, it will follow a predetermined path till it stops. It is clear the algorithm, that the robot should follow is defined by the starting point and the starting angle of the robot, with respect to any of the walls of the rectangle. Without loss of generality, we define the longer sides of the rectangle as the horizontal sides, and one of the horizontal sides we call the base or x-axis. One of the shorter, or the vertical, sides we denote as the y-axis. The intersection points or the corner of the base and the y-axis is called the origin point. The starting angle is measured with respect to the y-axis which is denoted by $\theta \in \mathbb{R}$. We always measure the angle in clockwise direction of the interior of the rectangle. It is clear that if the robot starts with an angle 0 or $\frac{\pi}{2}$ with respect to any wall, then the robot will either hit a corner or will go into an infinite oscillation along one line between two opposite walls. So it follows that $\theta \in \left(0, \frac{\pi}{2}\right)$. Without loss of generality we assume that the robot will start from some point at either x-axis or y-axis. We also measure the starting point of the robot from the origin along the y-axis and denote that with $y_0 \in \mathbb{R}$. If the robot starts at a point on the base x distance from the origin, then, from geometry we can say that $y_0 = -x \tan\theta$. So $y_0 \in [-b \cdot \tan\theta, b \cdot \tan\theta]$.

At any given time, the points on the perimeter where the robot has bounced form a set of "bounced points". It is to be noted that none of the bounced points can be part of the opening. Thus, if the distance between any two visited points are at most 1, then it is guaranteed to not have the opening between those two points. If the distance between any visited point and a corner are at most 1, then it is guaranteed to not have the opening in between the point and the corner. At any

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given time those parts of the perimeter which are guaranteed to not have the opening are called “discovered points”.

When the robot bounces on a vertical side of the rectangle it changes its horizontal direction. This process is called a “**Switch**”. Starting from one vertical side, when a robot moves, to another vertical side, it is called a “**Traversal**”. We do not consider the very first starting action as a switch. The vertical distance of the starting point from the base for a vertical side after t traversal is given y_t.

Related works are presented in Section II. In Section III we discuss our preliminary results, followed by Section IV with our main results and observations. We also present some additional interesting results in section V. Finally, the future work and conclusion are provided in Section VI.

II. Related Work

Robot path planning has been studied in the field of robotics by a wide range of researchers for many purposes, such as for *coverage path planning* where the goal is to cover all points in a given environment by the robot [2], [3], [4]. Coverage as well has several applications, such as lawn mowing and milling [5], [6].

Through analysis of some dynamical systems, bouncing robots were studied considering both random bounces and bounces in which the angle of reflection equals the angle of incident [7], [8]. Such dynamic systems also arise and are studied in classical mechanics [9]. The coverage problem for a given bouncing robot in a rectangular room, where the robot reflects with $\frac{\pi}{4}$ angle from the obstacle, was also studied in [1]. The topic is of interest in combinatorics and discrete math as well, where the bouncing robots refer to billiard balls or ergodic systems [10], [11].

The path on which the robot moves is the same as the path that a light ray follows if the ray can reflect from the edges of the polygon. As such, this research has a strong connection with the topic of visibility with reflection, which is studied in the field of computational geometry for specific cases: diffuse or specular. In specular reflection, the ray of light travels according to the rules of reflection where the angle of reflection equals the angle with which the light hits the obstacle. In diffuse reflection, the light reflects in all different directions after hitting the obstacle. Aronov et al. studied the combinatorial complexity of the light emanating from a given point in a given polygon P when one or more edges of P are reflective [12], [13]. Parsad et al. established a new bound on the complexity of the area that is visible for a given point after k diffuse reflections [14]. Ghosh et al. have proposed an algorithm to calculate the diffuse reflection path between two given points s and t [15]. O’Rourke and Petrovici also proposed a natural mirror trapping question where they explored if light can be trapped in a given environment, and developed some partial answers for the question [16]. Barequet et al. determined the maximum number of diffuse reflections needed for a point to illuminate the entirety of a simple polygon [17]. Recently, Eppstein studied the path of a ray traversing an octagonal mirror maze [18].

III. Preliminaries

In this section we discuss some of the preliminary results, that will be used for establishing our final results. One of the proof is described from our previous paper [1], which we are presenting here for the sake of completeness.

Lemma 1. *If the robot hits the same point twice on the boundary, then the robot’s path repeats after that.*

Proof. The sides of the rectangle are orthogonal to each other. Hence, from the laws of symmetric reflection, it follows that for every bounce on the wall, the angle of incidence and the angle of reflection on any vertical side is θ and $\left(\frac{\pi}{2} - \theta\right)$ for any horizontal side. Thus, if the robot hits a point twice then it will always use the same reflection angle on that point. Hence, from the same point, if the robot uses the same direction, then it arrives at the same set of points going further. Thus, the path repeats. □

It is evident that for finding the opening, we need an algorithm, the robot should discover new points so that the set of discovered points are increased. For this the robot should avoid the corners and make sure the all the bounced points are unique.

It follows that the two consecutive bounces on horizontal sides will be on two opposite horizontal sides and the horizontal distances between those two consecutive bounces are same and it is denoted by d, which is given by $d = a \tan\theta$. The distance between two consecutive bounce on the same horizontal side within the same traverse is $2d$.

Lemma 2. *The position y_t of the robot along the vertical side, measured from the base, after t traverse is given by*

$$y_t = y_0 + \frac{b \cdot t}{\tan\theta} - n_t \cdot a,$$

in which, $n_t = \lfloor \frac{y_0 + b \cdot t/\tan\theta}{a} \rfloor$.

Equivalently, $y_t = (y_0 + b \cdot t/\tan\theta) \mod a$.

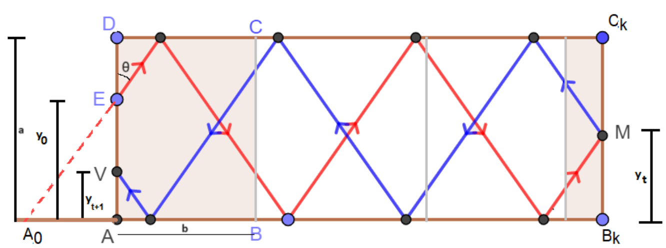

Fig. 2: Perpetual Bounce (not to scale)

Proof. This bouncing on the vertical side is equivalent to continuing in a straight line through the wall inside the unfolded rectangle, which can be created reflecting (unfolding) the rectangle about the wall. Refer to Figure 2. Thus, if the robot finishes t total traversals, then it would have gone through the t unfolded rectangles of size b, equivalent to 1 traversal of a rectangle of horizontal side $(t \cdot b)$. Within that

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rectangle of sides $a \times (t \cdot b)$ at every bounce, the robot moves $(a \cdot \tan\theta)$ distance away horizontally from the corner.

Starting from y_0 distance away from the base on the vertical side is equivalent to starting from $y_0 \cdot \tan\theta$ distance away from the left (sides opposite to the direction of the starting movement) of the origin on the horizontal side. From this, it follows that

$$y_0 \cdot \tan\theta + b \cdot t = n_t \cdot a \cdot \tan\theta + r_t,$$

in which

$$n_t = \left\lfloor \frac{y_0 \cdot \tan\theta + b \cdot t}{a \cdot \tan\theta} \right\rfloor,$$

and r_t is the remainder of length $(y_0 \cdot \tan\theta + b \cdot t)$ divided by $(a \cdot \tan\theta)$. It follows that, r_t equals

$$(y_0 \cdot \tan\theta + b \cdot t) \mod (a \cdot \tan\theta) = (y_0 \tan\theta + b \cdot t) \mod d.$$

Also, it follows $r_t = y_t \cdot \tan\theta$, and $r_0 = y_0 \cdot \tan\theta$. From there, the result of the lemma follows. ☐

From the above lemma the following observation follows.

Observation 1. *For k-th traversal*

1)
$$y_0 \cdot \tan\theta + b \cdot t = n_t \cdot d + r_t$$
$$y_0 \cdot \tan\theta + b \cdot (t-1) = n_{t-1} \cdot d + r_{t-1}$$
$$\implies r_t - r_{t-1} = b + (n_t - n_{t-1}) \cdot d$$
$$\implies r_t = (b + r_{t-1}) + (n_t - n_{t-1}) \cdot d.$$

So, $r_t = (b + r_{t-1}) \mod d$, in which r_{t-1} is the distance of the first bounce on any horizontal side from its starting vertical side, and r_t is the distance of the last bounce on the same horizontal side from its ending vertical side within a traverse.

2) *The i-th bounce on the base horizontal side is $(r_0 + i \cdot 2d)$ away from the starting vertical side.*

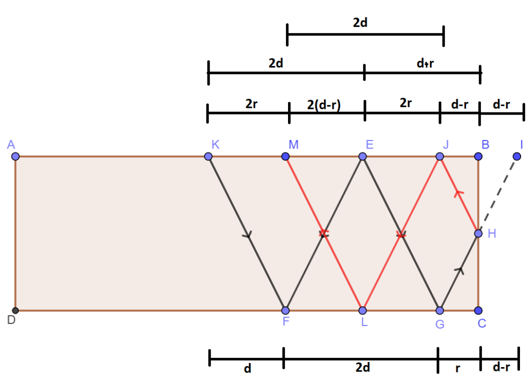

Fig. 3: Distance between visited points

Lemma 3. *In two consecutive traversals, the maximum distance between any two closest visited points on any horizontal side is $\max\{2r, 2(d-r)\}$, where r is the distance of the last bounce on the same horizontal side from the vertical side about which the switch occurs.*

Proof. Consider the robot hits a vertical side coming from a horizontal side, we denote that horizontal side by CD and the other horizontal side which is opposite to CD as AB. We denote the vertical side where the switch happens as BC. We denote the corner where BC and AB meet as B, and the corner where BC and CD meet as C.

The hitting point G on CD is at a distance r from the corner C. Hence, the last point E the robot hit on AB must be $r + d$ distance away from the corner B; see Figure 3.

After hitting the vertical side BC, the robot will move to the side AB and will hit a point J at $d - r$ distance from the corner. Thus, the distance between the points E and J is $(d+r) - (d-r) = 2r$, and the distance of the point E from the next visited point M after J following the switch on AB is $2d - 2r = 2(d-r)$. Therefore, the distance from any corner to the nearest visited point is less than $\max\{2r, 2(d-r)\}$. Furthermore, the distance from any corner on the switching side to the adjacent visited points is $\max\{r, (d-r)\}$. Let the first bounce from the starting vertical side be of distance r_0 from D, and $r_0 = d + (d - r)$. The distance from any corner to the nearest visited point is either r if $r_0 > r$, or $d - r$ otherwise. Therefore, the distance from any corner to the nearest visited point is $\max\{r, (d-r)\}$. Combining these, the maximum distance between the two nearest visited points after one switch is $\max\{2r, 2(d-r)\}$. ☐

Now we choose d in a particular way, such that

$$d = \frac{b}{p/q}, \tag{1}$$

in which

$$p, q \in \mathbb{Z}_+,\ gcd(p,q) = 1,\ p > q,$$

and b, p has no common divisor, and a, q has no common divisor. More precisely

$$\forall i \in \mathbb{Z}_+ : (i \cdot b) \mod p = 0 \implies i = k \cdot p, k \in \mathbb{Z}_+,$$
$$\forall i \in \mathbb{Z}_+ : (i \cdot a) \mod q = 0 \implies i = k \cdot q, k \in \mathbb{Z}_+.$$

So in this case $\frac{b}{d} = \frac{p}{q} > 1$, and $\forall i \in \{1, \ldots, q-1\}$, $(i \cdot p) \mod q > 0$. This implies that $\forall i \in \{1, \ldots, q-1\}$,

$$i \cdot p = \left\lfloor \frac{i \cdot p}{q} \right\rfloor \cdot q + (i \cdot p) \mod q,$$
$$\implies i \cdot p \cdot \frac{d}{q} = \left\lfloor \frac{i \cdot p}{q} \right\rfloor \cdot d + ((i \cdot p) \mod q) \cdot \frac{d}{q},$$
$$\implies i \cdot b = \left\lfloor \frac{i \cdot p}{q} \right\rfloor \cdot d + ((i \cdot p) \mod q) \cdot \frac{b}{p}.$$

That means $\forall i \in \{1, \ldots, q\}$,

$$(i \cdot b \mod d) = ((i \cdot p) \mod q) \cdot \frac{b}{p}.$$

Going forward, unless otherwise specified, we use this particular value of d to calculate the starting angle θ given by $\theta = \arctan\left(\frac{d}{a}\right)$.

For the sake of completeness, we reproduce the following lemma and the proof which was developed in [1].

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Lemma 4. *Consider $x, y \in \mathbb{Z}_+$ which are co-prime and $x < y$. Let R be the collection of all the remainders of $p \cdot y \mod x, \forall p \in \{1, \ldots, x\}$. Then, $R = \{0, \ldots, x-1\}$.*

Alternatively, we can rewrite this above lemma for two co-prime integer x, y. With $x < y$ all the remainders of $p \cdot y \mod x, \forall p \in \{1, \ldots, x\}$ are unique.

Thus, if the robot made the traversal q times then $(q \cdot b) \mod d = ((q \cdot p) \mod q) \cdot \frac{b}{p} = 0$. Thus, after q number of traversals there will be no remainder and the robot will hit a corner. After $k < q$ number of traversals, it follows that $((k \cdot p) \mod q) \cdot \frac{b}{p} > 0$, as b is not a multiple of p. Thus, there will be a positive remainder, and the robot will not hit a corner.

Moreover, the robot will encounter different remainders for every switch of every bouncing on the vertical side as per Lemma 4. Thus, every hit on the vertical sides the distances from the nearest corners will be unique before encountering a corner. In other words, the starting point for every traversal will be unique. Thus, the subsequent bouncing points will be also unique.

Lemma 5. *Starting from a corner if the robot first hits any corner, the maximum distance between the two adjacent visited points on any horizontal side is $\frac{2d}{q}$ and adjacent visited points are uniformly distributed over the range $(0, 2d]$.*

Proof. If the robot performs k number of traversals, then it is equivalent to one traversal within the rectangle with vertical side a and horizontal side $(k \cdot b)$.

When a robot bounces on the horizontal sides, in each bounce it moves d distance horizontally. Suppose the robot starts from a corner of a vertical side L and moves to the opposite vertical side R, which is $(k \cdot b)$ away from the side L. If the robot does not hit any corner of R, then the last bounce of the robot on a horizontal side just before it hits R will be some distance away from R. Let that distance be r, which is given by $r = (k \cdot b) \mod d$.

Suppose the robot starts from a corner and reaches another corner after q number of traversals. This is equivalent to traversals in the rectangles with vertical side a and horizontal sides $(k \cdot b), \forall k \in \{1, \ldots, q\}$, which will generate q number of different remainders or $q - 1$ number of unique non-zero remainders, as per Lemma 4. In every traversal, the maximum distance between two adjacent points will decrease by twice the changes in the remainder.

Each remainder corresponds to one unique integer in $\{1, \ldots, q\}$, which means that the remainders are uniformly distributed, over the range $(0, d]$. Hence, at the end when the robot hits the corner, the difference between the encountered remainders will be $\frac{d}{q}$. Now the remainders are equidistant over d, so the differences of the remainders with d will also be equidistant and will correspond to the other members of the remainder set. Thus, as per Lemma 3, the maximum distance between two adjacent visited points will be $2 \cdot \frac{d}{q}$ and will be uniformly distributed over the range $(0, 2d]$. □

Lemma 6. *If the robot starts from a corner, and after a sequence of bounces and traversals, upon encountering a corner, the maximum distance between the two adjacent visited points on any vertical side is $a / \lfloor \frac{q}{2} \rfloor$ or $a / \lceil \frac{q}{2} \rceil$.*

Proof. If the robot starts from a corner, when it first hits a corner it will have q number of traversals. Half of that number will bounce on one vertical side and the rest will be on another vertical side.

Consider a bouncing point O on a vertical side after k traversals. If the remainder after k traversals is r_k, then the point O is $r_k \cdot \tan\theta$ distance from the horizontal side from where it came to point O.

Now $\forall k \in \{1, \ldots, q\}, r_k$ will be unique and equidistant, which in turn indicates that the points on the vertical sides are also equidistant.

Furthermore, it means that in one vertical side there will be $\lfloor \frac{q}{2} \rfloor$ points and another vertical side there will be $\lceil \frac{q}{2} \rceil$ points, which are equally spread over the side of length a. Thus, the max distances between those points in one vertical side will be $a / \lfloor \frac{q}{2} \rfloor$ and in another vertical side will be $a / \lceil \frac{q}{2} \rceil$. □

Lemma 7. *Suppose the robot starts from a corner C of the given rectangle with one particular angle. The next time it hits a corner, that corner must not be C.*

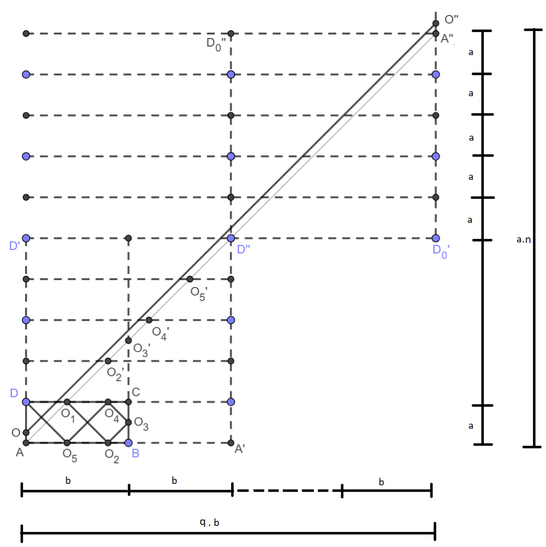

Fig. 4: Arithmetic Billiard Bounce

Proof. If the robot starts from a corner, it will hit a corner the first time after q traversals. The path followed by the reflection on a wall is equivalent to the straight path taken by the robot going through the wall on the rectangle that is created by the reflection of the actual rectangle about the wall (unfolding the rectangle about the wall). Refer to Figure 4. Here, the rectangle is reflected about the vertical side q times, and each of these reflected rectangles is again reflected about the horizontal side some number times, which we denote by n and given by $n = \frac{b \cdot q}{d} = p$. Here, unfolding of the rectangle occurs q times horizontally and p times vertically. For the robot to move from one corner to the same corner after q vertical and p horizontal bounces, both p and q must be even, which is not possible as p and q are co-prime. So starting

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from a corner, the robot after q traversals will go to a corner other than the starting corner. ☐

Lemma 8. *If the robot starts at a point on a side which is $y_0 \leq a$ distance from a corner C, then the robot returns to the starting point after $2 \cdot q$ traversals.*

Proof. The path followed by the reflection on a wall is equivalent to the straight path taken by the robot going through the wall on the rectangle that is created by the reflection of the actual rectangle about the wall (unfolding the rectangle about the wall). Refer to Figure 4. Here, the rectangle is reflected about the vertical side q times, and each of these reflected rectangles is again reflected about the horizontal side some number times, which we denote by n and is given by p. This way, all the combined unfolded rectangles will create a rectangle of side lengths $(b{\cdot}q)\times(a{\cdot}n)$. The robot goes from the starting corner to another corner of that rectangle along the diagonal, which creates an angle θ with the vertical side; see Lemma 7. If the robot is allowed to hypothetically continue after it first hits the corner with the same starting angle, then it will come back to the starting corner retracing the path. This is equivalent to moving through the rectangle of side lengths $(2 \cdot b \cdot q) \times (2 \cdot a \cdot p)$. We denote that rectangle as R.

If the robot starts at y_0 distance from the previous starting corner with the same angle θ, then it will be parallel to the diagonal of the rectangle R. It will finally hit the point on the vertical side that is equivalent to the same side of the starting vertical side at y_0 distance from the previous starting corner. Thus, the robot returns to the original position after $2 \cdot q$ traversals and $2 \cdot p$ vertical bounces. ☐

IV. Main Results

From the above discussion, it follows that if the robot starts at $y_0 \leq (a \mod (q \cdot x))$ distance from a corner, then the robot returns to the starting point after $2 \cdot q$ traversals. By that time, the distance between the visited points on the horizontal sides are $\frac{2d}{q}/2 = \frac{d}{q}$, and the maximum distances of the end visiting points from the corners are $\frac{d}{2 \cdot q}$. Whereas the distance between the visited points on the vertical sides are $(a/\left\lfloor \frac{q}{2} \right\rfloor)/2$ or $(a/\left\lceil \frac{q}{2} \right\rceil)/2$. These can be resolved to $\left\lceil \frac{a}{q} \right\rceil$ as the maximum gap. The maximum distances of the end visiting points from the corners are $(a \mod q)$. To ensure that the robot will be able to escape the rectangle, the maximum distance between two visited points must be at most the minimum length of the opening is 1. Therefore, the choice of d must satisfy:

$$\frac{b}{p} = \frac{d}{q} \leq 1,\ \frac{a}{q} \leq 1,\implies a \leq q,\ b \leq p,$$

This implies $\frac{b}{a} \leq \frac{p}{q}$.

Lemma 9. *Suppose the robot starts from a corner. The next time that it hits a corner, it has traveled distance $\left(\frac{b \cdot q}{\sin\theta}\right)$.*

Proof. The robot covers $(b \cdot q)$ when it starts from a corner and finishes in another corner. So, diagonally it covers total $\left(\frac{b \cdot q}{\sin\theta}\right)$ distance. ☐

If the robot starts at y_0 distance from the origin at an angle θ, it will travel distance

$$2 \cdot \left(\frac{b \cdot q}{\sin\theta}\right) = 2 \cdot b \cdot q \frac{\sqrt{a^2 + d^2}}{d} = 2\sqrt{a^2 \cdot p^2 + b^2 \cdot q^2}.$$

We can minimize this distance by carefully choosing the values of p, q.

Considering the above observations, we formulate an algorithm which ensures that the robot will find the opening of minimum length 1 within a rectangle of size $a \times b$ as follows: Input $a, b \in \mathbb{R}_+$

Choose
$$p, q \in \mathbb{Z}_+$$
such that
$$p > b, \quad q > a,$$
$$\gcd(p, q) = 1, \quad p > q,$$
$$\forall i \in \mathbb{Z}_+ : (i \cdot b) \mod p = 0 \implies i = k \cdot p, k \in \mathbb{Z}_+,$$
$$\forall i \in \mathbb{Z}_+ : (i \cdot a) \mod q = 0 \implies i = k \cdot q, k \in \mathbb{Z}_+.$$
Calculate
$$d = \frac{q}{p} \cdot b,$$
$$\theta = \arctan\left(\frac{d}{a}\right).$$
Choose
$$y_0 \leq (a \mod q).$$
Minimize
$$2\sqrt{a^2 \cdot p^2 + b^2 \cdot q^2}.$$
Output $y_0,\ \theta$.

It is to be noted that the condition $a \leq b$, that we assumed is important for the constraints $p > b, \quad q > a$, to be true. Another important observation is that the constraints $p > b, \quad q > a$, need to be strict inequalities; otherwise, the next two constraints will not always be correct.

V. Additional Results

The following results follow the case illustrated in Figure 5.

Observation 2. *By the time the robot covers the boundary of the given rectangle, the maximum radius of a circle that fits inside the rectangle and does not intersect the robot's path is $\left(\frac{d}{q}\right)\cos\theta$.*

Lemma 10. *There exists an algorithm that guarantees the robot will continue to bounce forever and cover the entire perimeter and interior of the rectangle.*

Proof. We choose $d < p$ such that $\frac{b}{d} \in \mathbb{R}\backslash\mathbb{Q}$ ($\frac{b}{d}$ is not a rational number), and let the robot start from a corner. $\forall t \in \mathbb{Z}_+$,

$$b \cdot t = \left\lfloor \frac{b \cdot t}{d} \right\rfloor \cdot d + r_t$$

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$$\implies \frac{b}{d}\cdot t = \left\lfloor \frac{b\cdot t}{d} \right\rfloor + \frac{r_t}{d}.$$

Here, $\frac{b}{d} \in \mathbb{R}\backslash\mathbb{Q}$, and $t \in \mathbb{Z}_+$. So $\frac{b}{d}\cdot t \in \mathbb{R}\backslash\mathbb{Q}$. So $\frac{r_t}{d} \in \mathbb{R}\backslash\mathbb{Q}$. In this case $\frac{r_t}{d} \neq 0$, which means $r_t \neq 0$, and therefore the robot will never hit any corner.
Now $\frac{b}{d}\cdot t \in \mathbb{R}\backslash\mathbb{Q}$ and the $\left(\frac{b}{d}\right) \mod 1 = \frac{r_1}{d}$. Per Weyl's Equidistribution Theorem, $\forall t \in \mathbb{Z}_+, \left(\frac{b}{d}\cdot t\right) \mod 1$ are equidistributed within the interval $[0,1]$. So, $\forall t \in \mathbb{Z}_+, \frac{r_t}{d}$ are equidistributed within the interval $[0,1]$, which in turn means $\forall t \in \mathbb{Z}_+, r_t$ equidistributed within the interval $[0,d]$. That means that the remainders $t\cdot(b \mod d), \forall t \in \mathbb{Z}_+$, are unique and their distances will diminish as t increases. When $t \to \infty$ then, $r_t = t\cdot(b \mod d) \to 0$. So the robot will never be in a loop, and the distances of the visited points on the perimeters will reduce, eventually reaching 0. As a result, the smallest unvisited circle inside the rectangle will be smaller and smaller, and will eventually be 0. Thus, the entire perimeter and the interior will eventually be visited. ☐

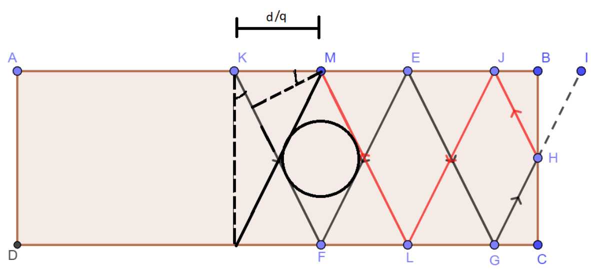

Fig. 5: Circle not visited

VI. Future Work and Conclusion

Given a rectangular room with an opening of a given length, we studied a bouncing robot that follows the billiard path trajectory. We produced a method to determine the starting point and starting angle for the robot based on the length of the room to be able to find the opening.

This work brings us closer to characterizing the movement of bouncing robots in more complex environments, and under different bouncing rules. This is useful in the design of ergodic robot systems, as studied in other works [1], [19], [20], [21], [22]. Specifically, the characteristics of bouncing robots movement in different environments, including parallelograms, convex polygons, or a set of connected rectangles are of key interest. One future work extending from this work will be to explore how the robot's trajectory changes under asymmetric bouncing rules.

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