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Minimum Wheel-Rotation Paths for Differential-Drive Mobile Robots

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***Abstract*— Characterizing optimal paths for mobile robots is an interesting, important, and challenging endeavor. Not only they are interesting with respect to the optimized criteria, but also they offer a family of motion primitives that can be used for motion planning in the presence of obstacles. This paper presents characterization of shortest paths for differential-drive mobile robots, with the goal of classifying solutions in the spirit of Dubins curves and Reeds-Shepp curves for car-like robots. To obtain a well-defined notion of *shortest*, the total amount of wheel rotation is optimized. Using Pontryagin maximum principle and other tools, we establish the existence of optimal trajectories, and derive the set of optimal paths. Some Reeds-Shepp curves appear in the set of optimal paths, whereas there are optimal paths which are different from Reeds-Shepp curves. To the best of our knowledge, this is the first progress on the problem.**

I. Introduction

This paper presents the family of all minimum wheel-rotation trajectories for differential-drive mobile robots in the plane without obstacles. By *wheel-rotation* we mean the distance travelled by the robot wheels, which is independent of the robot maximum speed. Thus, minimizing wheel-rotation is a natural variation of the shortest path problem by Dubins [4] and Reeds and Shepp [6], in which the distance travelled by the car is minimized. In this regard, this work has been basically motivated by Dubins and Reeds-Shepp shortest paths for car-like vehicles. The first work on shortest paths for car-like vehicles is done by Dubins [4]. He gives a characterization of time-optimal trajectories for a car with a bounded turn radius. In that problem, the car always moves forward with constant speed. He uses a purely geometrical method to characterize such shortest paths. Later, Reeds and Shepp [6] solve a similar problem in which the car is able to move backward as well. They identify 48 different shortest paths. Shortly after Reeds and Shepp, their problem is solved and also refined by Sussmann and Tang [13] with the help of optimal control techniques. Sussmann and Tang show that there are only 46 different shortest paths for Reeds-Shepp car. Souères and Laumond [11] classify the shortest paths for a Reeds-Shepp car into symmetric classes.

However, optimal trajectories for nonholonomic systems are interesting not only because of the criterion that they optimize, but also because they have a property that makes them useful for motion planning in the presence of obstacles. If we restrict the motions of the system to the set of optimal trajectories,

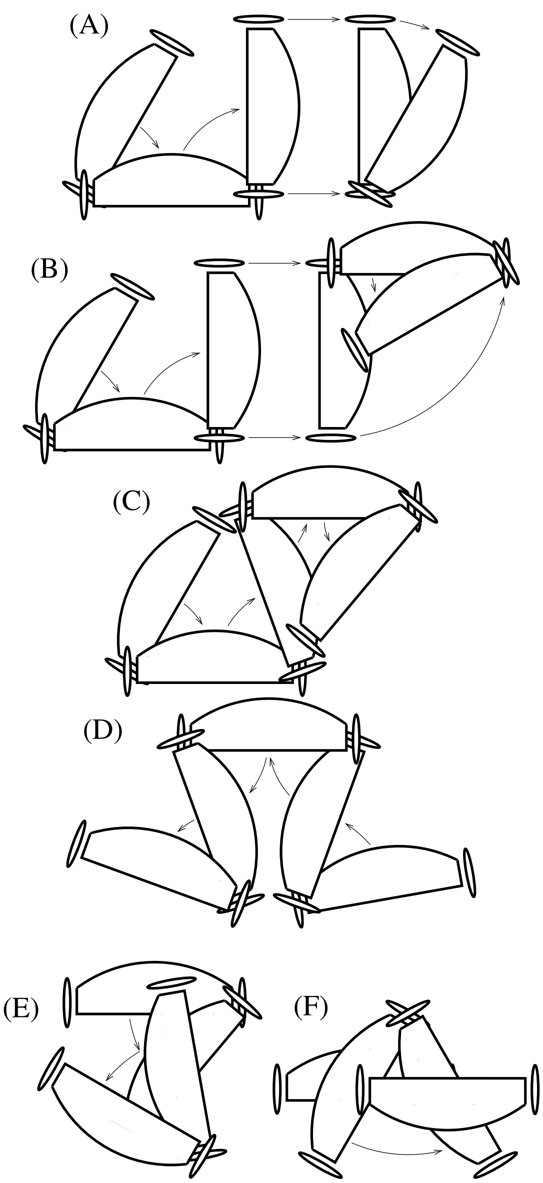

Fig. 1. Minimum wheel-rotation trajectories up to symmetry: (A) and (B) are composed of two swings, straight, and one or two swings respectively. (C) and (D) are composed of four alternating swings. (E) is composed of swing, rotation in place, and swing. (F) is composed of rotation in place, swing, and rotation in place.

we still conserve some important properties of the system such as *small-time local controllability*. Consequently, local planners that are based on families of optimal trajectories satisfy the *topological property* [9]. Hence, different families of optimal trajectories provide local planners that can be helpful in different applications.

In [1], the time-optimal trajectories for the differential drive is studied, and a complete characterization of all time-optimal trajectories is given. Time-optimal trajectories for the differential drive consist of rotation in place and straight line segments. Our paper shows that minimum wheel-rotation trajectories exist for all pairs of initial and goal configurations; and they are different from time-optimal trajectories. They are composed of rotation in place, straight line, and swing segments (one wheel stationary and the other rolling). Since any subpath of an optimal path is optimal, 28 different minimum wheel-rotation trajectories are identified in our work, which are maximal with respect to subpath partial order.

Souères and Boissonnat [10] study the time optimality of Dubins car with angular acceleration control. They present an incomplete characterization of time-optimal trajectories for their system. However, full characterization of such time-optimal trajectories seems to be difficult because Sussmann [12] proves that there are time-optimal trajectories for that system that require infinitely many input switchings (chattering or Fuller phenomenon). Sussmann uses Zelikin and Borisov theory of chattering control [14] to prove his result. Chyba and Sekhavat [3] study time optimality for a mobile robot pulling one trailer. For a numerical approach to time optimality for differential-drive robots see Reister and Pin [7]. For a study on acceleration-driven mobile robots, see Renaud and Fourquet [8].

The approach that we use to derive optimal trajectories is similar to the one used in [1], [3], [10], [13]. However, the difference between our method and the aforementioned methods is that we develop specific geometric methods to rule out non-optimal trajectories. We first prove that minimum wheel-rotation trajectories exist for our problem. It is then viable to apply the necessary condition of the Pontryagin Maximum Principle (PMP) [5]. The geometric interpretation of the PMP leads to geometric arguments that rule out some non-optimal trajectories. The remaining finite set of candidates are compared with each other to find the optimal ones. Some of the proofs of theorems, lemmas, and propositions are omitted due to space limitations.

II. Problem Formulation

A differential-drive robot [1] is a three-dimensional system with its configuration variable denoted by $q = (x, y, \theta) \in \mathcal{C} = \mathbb{R}^2 \times \mathbb{S}^1$ in which x and y are the coordinates of the point on the axle, equidistant from the wheels, in a fixed frame in the plane, and $\theta \in [0, 2\pi)$ is the angle between x-axis of the frame and the robot local longitudinal axis (see Figure 2).

The robot has independent velocity control of each wheel. Assume that the wheels have equal bounds on their velocity. More precisely, $u_1, u_2 \in [-1, 1]$, in which the inputs u_1 and u_2 are respectively the left and the right wheel velocities, and the input space is $U = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$. The system is

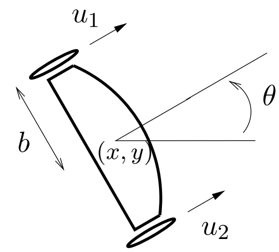

Fig. 2. Differential-drive model

$$\dot{q} = f(q, u) = u_1 f_1(q) + u_2 f_2(q) \tag{1}$$

in which f_1 and f_2 are vector fields in the tangent bundle $T\mathcal{C}$ of configuration space. Let the distance between the robot wheels be $2b$. In that case,

$$f_1 = \frac{1}{2}\begin{pmatrix}\cos\theta \\ \sin\theta \\ -\frac{1}{b}\end{pmatrix} \text{ and } f_2 = \frac{1}{2}\begin{pmatrix}\cos\theta \\ \sin\theta \\ \frac{1}{b}\end{pmatrix}. \tag{2}$$

The Lagrangian L and the cost functional J to be minimized are

$$L(u) = \frac{1}{2}(|u_1| + |u_2|) \tag{3}$$

$$J(u) = \int_0^T L(u(t))dt. \tag{4}$$

The factor $\frac{1}{2}$ above helps to simplify further formulas, and does not alter the optimal trajectories.

For every pair of initial and goal configurations, we seek an admissible control, i.e. a measurable function $u : [0, T] \to U$, that minimizes J while transferring the initial configuration to the goal configuration. Since the cost J is invariant by scaling the input within U, we can assume without loss of generality that the controls are either constantly zero ($u \equiv (0, 0)$) or saturated at least in one input, i.e. $\max(|u_1(t)|, |u_2(t)|) = 1$ for all $t \in [0, T]$. Throughout this paper, a trajectory for which $u \equiv (0, 0)$ over its time interval is called *motionless*.

III. Existence of Optimal Trajectories

As in [1], the system is controllable. Moreover, it can be shown that the system is small-time local controllable. Hence, there exists at least one trajectory between any pair of initial and goal configurations, and it is meaningful to discuss the existence of optimal trajectories. In the following, we will use a version of Filippov Existence Theorem to prove the existence of optimal trajectories.

Let the initial configuration be $q_0 = (x_0, y_0, \theta_0)$ and the goal configuration be $q_1 = (x_1, y_1, \theta_1)$. Let A_T be a set of configurations $A_T = B_T(x_0, y_0) \times \mathbb{S}^1 \subset \mathcal{C}$, in which T is an arbitrary positive real number, and $B_T(x_0, y_0)$ is the closed ball of radius T around (x_0, y_0) in the plane. The projection of robot configuration onto x-y plane cannot leave $B_T(x_0, y_0)$ in time T because $\sqrt{\dot{x}^2 + \dot{y}^2} \leq 1$. Note that T here is both maximum time and the radius of $B_T(x_0, y_0)$. Assume T is large enough so that $(x_1, y_1) \in B_T(x_0, y_0)$. Let $G = \{(q_0, q_1)\} \subset \mathcal{C} \times \mathcal{C}$ be the pair of initial and goal configurations, and let Ω_{A_T} be the set of all admissible trajectory-control pairs $(q(t), u(t))$ defined on $[0, T]$ that transfer q_0 to q_1 while staying in A_T, i.e. $(q(0), q(T)) \in G$, and $q([0, T]) \subset A_T$. Furthermore, assume T is chosen such that $\Omega_{A_T} \neq \emptyset$. Define $Q(q) \subset \mathbb{R} \times T_q\mathcal{C} \cong \mathbb{R}^4$ as

$$Q(q) = \{(z_0, z) \mid \exists u \in U : z_0 \geq L(u) \text{ and } z = f(q, u)\}. \tag{5}$$

Finally, let $M_T = A_T \times U \subset \mathcal{C} \times \mathbb{R}^2$. The following remarks are necessary to verify applicability of the Filippov Existence Theorem, which will shortly be used:

1) Since the differential drive is controllable, there exists a trajectory $q_\tau(t)$ that transfers the initial configuration to the goal configuration in some time τ. Choosing $T = \tau$ guarantees that q_τ stays in A_T because $\sqrt{\dot{x}^2 + \dot{y}^2} \leq 1$. Thus, for every (q_0, q_1) the number T satisfying assumptions above actually exists, and in particular, the class Ω_{A_T} is nonempty for an appropriate choice of T.
2) The sets A_T, G, U, and M_T are compact and consequently closed. Also, $L(u)$ given by (3) is continuous on U, and $f(q, u)$ is continuous on M_T.
3) $Q(q)$ is convex for all q. Since U is convex and $f(q, \cdot)$ is a linear transformation, $f(q, U)$ is also convex. The fact that $L(\cdot)$ is a convex function helps to establish the claim.

Theorem 1 (Filippov Existence Theorem [2]). *Let A_T be compact, G closed, M_T compact, $L(u)$ continuous on U, and f continuous on M_T. Assume that $Q(q)$ are convex for all $q \in A_T$, and Ω_{A_T} is nonempty. The functional J has an absolute minimum in the nonempty class Ω_{A_T}, which is the set of all admissible trajectory-control pairs defined on $[0, T]$ that transfer the initial configuration to the goal configuration while staying in A_T.*

From this we derive the following corollary which establishes the existence of minimum wheel-rotation trajectories for the system described in (1).

Corollary 1. *Minimum wheel-rotation trajectories for the differential-drive exist.*

Proof. Observe that $\frac{1}{2} \leq L \leq 1$ along any trajectory because at least one input is saturated. Let T be chosen so that $\Omega_{A_T} \neq \emptyset$. Theorem 1 guarantees the existence of a minimum wheel-rotation trajectory-control pair $(q_T(t), u_T(t))$ in Ω_{A_T}. Let $J_T = J(u_T)$, and let τ be the time of q_T. In that case, $\tau \leq T$ because $(q_T(t), u_T(t))$ is in Ω_{A_T}. The observation helps to show that $J_T \leq \tau \leq T$. Using Theorem 1, $\Omega_{A_{2T}}$ contains a minimum wheel-rotation trajectory-control pair $(q_{2T}(t), u_{2T}(t))$. Let $J_{2T} = J(u_{2T})$. Note that $J_{2T} \leq J_T \leq T$ because all elements of Ω_{A_T} are contained in $\Omega_{A_{2T}}$. Any trajectory-control pair that is not in $\Omega_{A_{2T}}$ takes at least $2T$ time. The observation again helps to show that the cost of any trajectory-control pair that is not in $\Omega_{A_{2T}}$ is at least $2T/2 = T \geq J_{2T}$. Thus, $q_{2T}(t)$ is an absolute minimum wheel-rotation trajectory over all trajectories. ■

IV. Necessary Conditions

Since we proved the existence of optimal trajectories in the previous section, it is viable now to apply the Pontryagin Maximum Principle (PMP) which is a necessary condition for optimality.

A. Pontryagin Maximum Principle

Let the Hamiltonian $H : \mathbb{R}^3 \times \mathcal{C} \times U \to \mathbb{R}$ be

$$H(\lambda, q, u) = \langle \lambda, \dot{q} \rangle + \lambda_0 L(u) \tag{6}$$

in which λ_0 is a constant. According to the PMP [5], for every optimal trajectory $q(t)$ defined on $[0, T]$ and associated with control $u(t)$, there exists a constant $\lambda_0 \leq 0$ and an absolutely continuous vector-valued adjoint function $\lambda(t)$, that is nonzero if $\lambda_0 = 0$, with the following properties along the optimal trajectory:

$$\dot{\lambda} = -\frac{\partial}{\partial q} H, \tag{7}$$

$$H(\lambda(t), q(t), u(t)) = \max_{z \in U} H(\lambda(t), q(t), z), \tag{8}$$

$$H(\lambda(t), q(t), u(t)) \equiv 0. \tag{9}$$

Def 1. An *extremal* is a trajectory $q(t)$ that satisfies the conditions of the PMP. Also, an extremal for which $\lambda_0 = 0$ is called *abnormal*.

Let the switching functions be

$$\varphi_1 = \langle \lambda, f_1 \rangle \text{ and } \varphi_2 = \langle \lambda, f_2 \rangle, \tag{10}$$

in which f_1 and f_2 are given by (2). We rewrite (6) as $H = u_1\varphi_1 + u_2\varphi_2 + \lambda_0 L$. The PMP implies that an optimal trajectory is also an *extremal*; however, the converse is not necessarily true. Throughout the current section, we characterize *all* extremals because the optimal trajectories are among them. In the following sections, we will provide more restrictive conditions for optimality and we will rule out all non-optimal ones.

B. Switching Structure Equations

Lemma 1 (Sussmann and Tang [13]). *Let f_k be a smooth vector field in the tangent bundle of the configuration space $T\mathcal{C}$, and let $q(t)$ be an extremal associated with control $u(t)$ and adjoint vector $\lambda(t)$. Let φ_k be defined as $\varphi_k(t) = \langle \lambda(t), f_k(q(t)) \rangle$. It follows that*

$$\dot{\varphi}_k = u_1 \langle \lambda, [f_1, f_k] \rangle + u_2 \langle \lambda, [f_2, f_k] \rangle. \tag{11}$$

Lemma 1 reveals valuable information by relating the structure of the Lie algebra to the structure of φ_i functions. To complete the Lie closure of $\{f_1, f_2\}$, we introduce f_3 as the Lie bracket of f_1 and f_2:

$$f_3 = [f_1, f_2] = \frac{1}{2b} \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}. \tag{12}$$

Let $\varphi_3(t) = \langle \lambda(t), f_3(q(t)) \rangle$ be the switching function associated with f_3. Lemma 1 implies the structure of switching functions as follows [1]:

$$\dot{\varphi}_1 = -u_2\varphi_3, \dot{\varphi}_2 = u_1\varphi_3, \dot{\varphi}_3 = \frac{1}{4b^2}(-u_1+u_2)(\varphi_1+\varphi_2). \tag{13}$$

Note that f_i's are linearly independent. Consequently, $\{f_1(q), f_2(q), f_3(q)\}$ forms a basis for $T_q\mathcal{C}$. As an immediate consequence of the PMP and Lemma 1, the following proposition holds.

Proposition 1. *An abnormal extremal is motionless.*

Proof. If $\lambda_0 = 0$, then (9) implies $u_1\varphi_1 + u_2\varphi_2 \equiv 0$. This means $|\varphi_1| \equiv |\varphi_2| \equiv 0$ because by maximization of the Hamiltonian, we must have $u_i\varphi_i = |\varphi_i|$ for $i = 1, 2$. For a detailed argument, see [1]. Consequently, φ_1 and φ_2 are constantly zero, and $\dot{\varphi}_1 \equiv \dot{\varphi}_2 \equiv 0$. In this case, $|\varphi_1| + |\varphi_2| + |\varphi_3| \neq 0$ because $\{f_1, f_2, f_3\}$ forms a basis for tangent space of the configuration space, and φ_i's are the coordinates of a nonzero vector $\lambda(t)$ in this basis. Thus, $\varphi_3 \neq 0$ and (13) imply $u_1 \equiv u_2 \equiv 0$. ■

C. Extremals

Having dealt with abnormal extremals in Proposition 1, we may now, without loss of generality, scale the Hamiltonian (6) so that $\lambda_0 = -2$. More precisely, the PMP conditions are valid if we replace $\lambda(t)$ by $-\frac{2\lambda(t)}{\lambda_0}$ and λ_0 by -2 in (6). We will assume that $\lambda_0 = -2$ for the rest of the paper. In that case, the Hamiltonian has the simple form

$$H = u_1\varphi_1 + u_2\varphi_2 - (|u_1| + |u_2|). \tag{14}$$

Equation 7 can be solved for λ to obtain

$$\lambda(t) = \begin{pmatrix} c_1 \\ c_2 \\ c_1y - c_2x + c_3 \end{pmatrix}, \tag{15}$$

in which c_1, c_2, and c_3 are constants. Let $i, j \in \{1, 2\}$ throughout the rest of the paper.

Def 2. For some $i = 1, 2$ an extremal for which $|\varphi_i(t)| = 1$ over some interval of time of positive length is called *singular*.

In Lemma 2, we will show that a non-singular extremal is motionless. We will also show that there are two categories of singular extremals depending on whether or not $c_1^2 + c_2^2 = 0$. The first category corresponds to $c_1^2 + c_2^2 \neq 0$, and consists of all singular extremals that are composed of a number of swing ($u_i = 0$) and straight ($u_1 = u_2$) intervals. Such extremals will be called *tight*. The second category corresponds to $c_1^2 + c_2^2 = 0$. Such extremals will be called *loose*.

Lemma 2. *Let $q(t)$ be an extremal associated with the control $u(t) = (u_1(t), u_2(t))$, adjoint vector function $\lambda(t)$, and switching functions $\varphi_i(t)$. Moreover, assume $q(t)$ is not motionless. In that case, the following hold:*

(*i*) $|\varphi_i(t)| \leq 1$.

(*ii*)

$$u_i(t) \in \begin{cases} [0,1] & \textit{if} \quad \varphi_i(t) = 1 \\ \{0\} & \textit{if} \quad |\varphi_i(t)| < 1 \\ [-1,0] & \textit{if} \quad \varphi_i(t) = -1 \end{cases}. \tag{16}$$

(*iii*) *If $c_1^2 + c_2^2 \neq 0$ and $|\varphi_1| = |\varphi_2| = 1$ over some interval $[t_1, t_2]$, then $u_1 = u_2$, and $\varphi_1 = \varphi_2$.*

(*iv*) *If $c_1^2 + c_2^2 \neq 0$ and $|\varphi_j| < |\varphi_i| = 1$ over a time interval $[t_1, t_2]$, then $u_j = 0$ and $|u_i| = 1$, in which $j \neq i$.*

(*v*) *If $c_1 = c_2 = 0$, then $\varphi_1 \equiv -\varphi_2$, and $u_1u_2 \leq 0$. In other words, the wheels move in opposite directions.*

Proof. (*i*) By inspection of (14), if $|\varphi_i| > 1$, there exist feasible controls yielding $H > 0$. This contradicts the maximum principle (8) and (9), which states that the maximum of H is zero.

(*ii*) If $|\varphi_i| < 1$, then (8) and (14) implies $u_i = 0$. In a similar way, if $\varphi_i = 1$, then $u_i \in [0, 1]$, and if $\varphi_i = -1$, then $u_i \in [-1, 0]$.

(*iii*) Assume $\varphi_1 = -\varphi_2$. From (2), (10), and (15) it follows that $c_1 \cos\theta + c_2 \sin\theta \equiv 0$. Differentiate this equation to obtain $\dot{\theta} \equiv 0$ because $-c_1 \sin\theta + c_2 \cos\theta \neq 0$. Thus, $2b\dot{\theta} = u_1 - u_2 = 0$, and (16) implies $u_1 = u_2 = 0$, which is not possible because $q(t)$ is not motionless.

(*iv*) This follows from (16).

(*v*) In that case, $\varphi_1 \equiv -\varphi_2$ by (2), (10), and (15). It follows from (16) that $u_1u_2 \leq 0$. ■

Geometric interpretation of tight extremals in Section IV-D will help to show that the number of switchings along a tight extremal is finite. Along a tight extremal we can assume $u_1 = 0, u_2 \in \{1, -1\}$ or $u_1 \in \{1, -1\}, u_2 = 0$ on swing segments, and $u_1 = u_2 \in \{1, -1\}$ on straight segments because at least one of the inputs is saturated. Thus, inputs are always either zero or bang $u_i \in \{1, 0, -1\}$ along tight extremals. In Section V-C, we will show that there may exist many wheel-rotation equivalent loose extremals, and for an appropriate choice of representative loose extremals, the inputs are always either zero or bang. In this section, we finished an elementary characterization of extremals. We have identified three main types of extremals:

1) *non-singular*: $u_1 \equiv u_2 \equiv 0$ (i.e. motionless)
2) *tight singular*: composed of a finite number of swing and straight segments
3) *loose singular*: $u_1u_2 \leq 0, \varphi_1 \equiv -\varphi_2$, and $|\varphi_1| \equiv |\varphi_2| \equiv 1$.

D. Geometric Interpretation of Tight Extremals

Let (x_1, y_1) and (x_2, y_2) be the coordinates of the left and the right wheel respectively. In that case,

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x - b\sin\theta \\ y + b\cos\theta \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x + b\sin\theta \\ y - b\cos\theta \end{pmatrix}. \tag{17}$$

Define functions $\gamma_1(x, y)$ and $\gamma_2(x, y)$ as

$$\gamma_1(x, y) = c_1y - c_2x + c_3 - 2b, \tag{18}$$

$$\gamma_2(x, y) = c_1y - c_2x + c_3 + 2b. \tag{19}$$

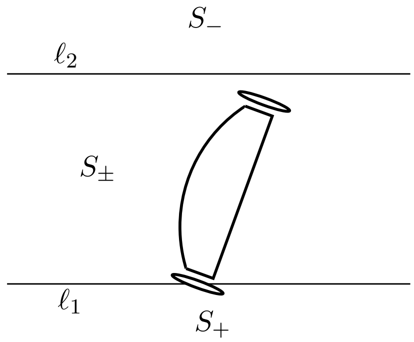

Fig. 3. The robot stays between two lines ℓ_1 and ℓ_2 along a tight extremal.

Taking (2), (10), (15), (17), (18), and (19) into account, we obtain

$$\varphi_1 = -\frac{1}{2b}\gamma_2(x_2, y_2) + 1 = -\frac{1}{2b}\gamma_1(x_2, y_2) - 1, \quad (20)$$

$$\varphi_2 = \frac{1}{2b}\gamma_1(x_1, y_1) + 1 = \frac{1}{2b}\gamma_2(x_1, y_1) - 1. \quad (21)$$

Note that $c_1^2 + c_2^2 > 0$, and consider the parallel lines $\ell_1 : \gamma_1(x, y) = 0$ and $\ell_2 : \gamma_2(x, y) = 0$ in the robot x-y plane. The value of γ_i at each point $P \in \mathbb{R}^2$ determines $d(P, \ell_i)$ scaled by $\sqrt{c_1^2 + c_2^2}$ for $i = 1, 2$, in which $d(P, \ell)$ is the signed distance of point P from a line $\ell \subset \mathbb{R}^2$. Since the base distance b of the robot is positive, $\gamma_2 > \gamma_1$ everywhere in the plane. Thus, ℓ_1 and ℓ_2 cut the plane into five disjoint subsets (see Figure 3): S_+, ℓ_1, $S_\pm$, ℓ_2, and S_- in which

$$S_+ = \{(x, y) \in \mathbb{R}^2 | \ \ \gamma_2(x, y) > \gamma_1(x, y) > 0\} \quad (22)$$

$$S_\pm = \{(x, y) \in \mathbb{R}^2 | \ \ \gamma_2(x, y) > 0 > \gamma_1(x, y)\} \quad (23)$$

$$S_- = \{(x, y) \in \mathbb{R}^2 | \ \ 0 > \gamma_2(x, y) > \gamma_1(x, y)\}. \quad (24)$$

Using Lemma 2 and (20) and (21), along a tight extremal $\gamma_1(x_i, y_i) \leq 0 \leq \gamma_2(x_i, y_i)$ for $i = 1, 2$. Thus, the robot always stays in the band $\ell_1 \cup S_\pm \cup \ell_2$ (see Figure 3). By appropriately substituting in (16), we obtain

$$u_1 \in \begin{cases} [-1, 0] & \text{if} \quad \text{wheel 2} \in \ell_1 \\ \{0\} & \text{if} \quad \text{wheel 2} \in S_\pm \\ [0, 1] & \text{if} \quad \text{wheel 2} \in \ell_2 \end{cases} \quad (25)$$

$$u_2 \in \begin{cases} [0, 1] & \text{if} \quad \text{wheel 1} \in \ell_1 \\ \{0\} & \text{if} \quad \text{wheel 1} \in S_\pm \, . \\ [-1, 0] & \text{if} \quad \text{wheel 1} \in \ell_2 \end{cases} \quad (26)$$

V. Characterization of Extremals

A. Symmetries

Assume $(q(t), u(t))$ is a minimum wheel-rotation trajectory-control pair that is defined on $[0, T]$. Let $\tilde{q}(t)$ be the trajectory associated with control $u(T-t)$, $\bar{q}(t)$ the trajectory associated with control $-u(t)$, and $\hat{q}(t)$ the trajectory associated with control $\hat{u}(t) = (u_2(t), u_1(t))$. Define the operators $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ acting on trajectory-control pairs by

$$\mathcal{O}_1 : (q(t), u(t)) \mapsto (\tilde{q}(t), u(T-t)) \quad (27)$$

$$\mathcal{O}_2 : (q(t), u(t)) \mapsto (\bar{q}(t), -u(t)) \quad (28)$$

$$\mathcal{O}_3 : (q(t), u(t)) \mapsto (\hat{q}(t), \hat{u}(t)). \quad (29)$$

Due to symmetries, $\mathcal{O}_1(q(t), u(t))$, $\mathcal{O}_2(q(t), u(t))$, and $\mathcal{O}_3(q(t), u(t))$ are also minimum wheel-rotation trajectories. $\mathcal{O}_1$ corresponds to reversing the extremal in time, $\mathcal{O}_2$ corresponds to reversing the inputs, and $\mathcal{O}_3$ corresponds to exchanging the left and the right wheels.

B. Characterization of Tight Extremals

In the following we give only the representatives of symmetric families of tight extremals. We will use L, R, and S to denote swing around the left wheel, the right wheel, and straight line motions, respectively. In cases where the directions must be specified, we use a superscript: $-$ is clockwise, $+$ is counter-clockwise, $+$ is forward, and $-$ is backward. Otherwise, the direction of swing is constant throughout the extremal. The symbol * means zero or more copies of the base expression. Subscripts are non-negative angles.

Depending on the distance between ℓ_1 and ℓ_2 we identify three different types of tight extremals.

Case 1: Let $d(\ell_1, \ell_2) = 2b$. Besides swing, the robot can move straight forward and backward by keeping the wheels on ℓ_i's. In this case, the extremals are composed of a sequence of swing and straight segments. In general, there can be an arbitrary number of swing and straight segments. Since the straight segments can be translated and merged together, a representative subclass with only one straight segment is described by the following forms:

- $(\mathbf{R}_\pi^- \mathbf{L}_\pi^-)^* \mathbf{R}_{\frac{\pi}{2}}^- \mathbf{S}^+ \mathbf{R}_{\frac{\pi}{2}}^- (\mathbf{L}_\pi^- \mathbf{R}_\pi^-)^*$
- $(\mathbf{R}_\pi^- \mathbf{L}_\pi^-)^* \mathbf{R}_{\frac{\pi}{2}}^- \mathbf{S}^+ \mathbf{L}_{\frac{\pi}{2}}^+ (\mathbf{R}_\pi^+ \mathbf{L}_\pi^+)^*$.

For optimal representatives of this class see (A) and (B) in Figure 1. We call such tight extremals *type I*.

Case 2: Let $d(\ell_1, \ell_2) > 2b$. The robot cannot move straight because it cannot keep the wheels on ℓ_i's over some interval of time. Thus, such extremals are of the form $(\mathbf{R}_\pi \mathbf{L}_\pi)^*$. Note that these extremals are subpaths of type I extremals.

Case 3: Let $d(\ell_1, \ell_2) < 2b$. In this case, the extremals are of the form $(\mathbf{L}_\gamma^- \mathbf{R}_\gamma^- \mathbf{L}_\gamma^+ \mathbf{R}_\gamma^+)^*$ in which $\gamma \leq \frac{\pi}{2}$. For optimal representatives of this class see (C) and (D) in Figure 1. We call such tight extremals *type II*.

Lemma 3. *Let $q(t)$ be a tight extremal associated with the control $u(t)$ that transfers (x_0, y_0, θ_0) to (x_1, y_1, θ_1). In this case*

$$J(u) = l = \int_0^T (\sqrt{\dot{x}^2 + \dot{y}^2}) dt, \quad (30)$$

i.e. the cost $J(u)$ is the length of the projection of $q(t)$ onto the x-y plane.

Proof. Since $2\sqrt{\dot{x}^2+\dot{y}^2} = \sqrt{(u_1+u_2)^2} = |u_1+u_2|$, it is enough to show $|u_1+u_2| = |u_1|+|u_2|$ along a tight extremal. Tight extremals are composed of swing and straight segments. Over a swing segment one of the inputs is zero, for instance $u_1 = 0$ in which case $|u_1+u_2| = |u_2| = |u_1|+|u_2|$. Over a straight segment $u_1 = u_2$ and $|u_1+u_2| = 2|u_1| = |u_1|+|u_2|$. ■

C. Characterization of Loose Extremals

So far, the only constraint on loose extremals is that the inputs must belong to intervals $[-1,0]$ or $[0,1]$, and they must have opposite signs. Thus, a variety of non-bang-bang controls generate various loose extremals. For instance, it can be verified that rotation round any point on the axle is a minimum wheel-rotation trajectory. In this section, we will first show that loose optimal trajectories can only cover a bounded region of the configuration space around the initial configuration. There may be different loose extremals that transfer the initial configuration to the goal configuration. In particular, there may exist different such loose extremals which have equal wheel rotation. Equivalence of wheel rotation defines equivalence classes of loose extremals. We will show in Lemma 6 that there exist a representative, that is composed of rotation in place and swing segments with a known structure, in every equivalence class.

Lemma 4. *Let $q(t)$ be a loose extremal associated with the control $u(t)$, and let ϑ be the length of the projection of $q(t)$ onto $\mathbb{S}^1$; in other words,*

$$\vartheta = \int_0^T |\dot{\theta}| dt. \tag{31}$$

In this case we have $J(u) = b\vartheta$.

Proof. Since $2b|\dot{\theta}| = |u_1 - u_2|$, it is enough to show that $|u_1-u_2| = |u_1|+|u_2|$ along a loose extremal. According to Lemma 2, $u_1u_2 \leq 0$ along a loose extremal. Thus, $|u_1u_2| = -u_1u_2$ which means $(|u_1|+|u_2|)^2 = (u_1-u_2)^2$. It is obvious then that $|u_1|+|u_2| = |u_1-u_2|$. ■

Lemma 5. *Let $(q(t),u(t))$ be a loose trajectory-control pair that tranfers the initial configuration (x_0,y_0,θ_0) to the goal configuration (x_1,y_1,θ_1). It follows that $J(u) = b|\theta_1-\theta_0+2k\pi|$ for some integer k. Furthermore, if $q(t)$ is optimal, then $J(u) \leq 5b\pi$.*

Proof. According to Lemma 4, the cost of a loose extremal is $b\vartheta$, in which ϑ is (31). In this case, $\vartheta = |\theta_1-\theta_0+2k\pi|$ for some integer k and the cost is $J(u) = b|\theta_1-\theta_0+2k\pi|$. For the second part, suppose $q(t)$ is optimal while $|\theta_1-\theta_0+2k\pi| > 5\pi$. It can geometrically be shown that $\sqrt{(x_1-x_0)^2+(y_1-y_0)^2} \leq 2bm$, in which m is an integer that satisfies the inequality $(m-1)\pi < |\theta_1-\theta_0+2k\pi| \leq m\pi$. Since $|\theta_1-\theta_0+2k\pi| > 5\pi$, we have $m \geq 6$. The cost of the trivial trajectory which is composed of rotation in place, going straight, and again rotation in place is not more than $2bm+b\pi$. Thus, we have $J(u) = b|\theta_1-\theta_0+2k\pi| > b(m-1)\pi > 2bm+b\pi$ because $m \geq 6$. This is contradictory to the optimality of $q(t)$. ■

Corollary 2. *Starting from an initial configuration, loose optimal trajectories are of bounded cost and bounded reach in the x-y plane. We call such optimal extremals* type III.

Lemma 6. *Let $(q(t),u(t))$ be a loose optimal trajectory-control pair that tranfers the initial configuration q_0 to the goal configuration q_1. There exists a trajectory-control pair $(\check{q}(t),\check{u}(t))$ transferring q_0 to q_1, in which $\check{u}$ is composed of a sequence of alternating rotation in place and swing segments in the same direction. Furthermore, $q(t)$ and $\check{q}(t)$ have the same wheel rotation, i.e. $J(u) = J(\check{u})$.*

Sketch of proof. Look at the time-optimal trajectories for the system described in (1) with $u_1 \in [-1,0], u_2 \in [0,1]$ (our claim for the case in which $u_1 \in [0,1], u_2 \in [-1,0]$ follows from a similar argument). We know the time-optimal trajectories for this modified system exist because its input space is convex. Upon applying the PMP with the time as the cost functional, the extremals are composed of a sequence of rotation in place and swing segments. Let $(\check{q}(t),\check{u}(t))$ be the time optimal trajectory-control pair, i.e. $\check{u}$ is composed of a sequence of rotation in place and swing segments. Lemma 4 implies that $J(u) = b\vartheta$ and $J(\check{u}) = b\check{\vartheta}$, in which ϑ and $\check{\vartheta}$ are as in (31). Since $\vartheta \equiv \pm\check{\vartheta}$ up to a multiple of 2π, and Lemma 5 holds for $(q(t),u(t))$, we have $J(u) = J(\check{u})$ because otherwise, it can be verified that $\check{u}$ is not time optimal. ■

VI. Minimum Wheel-Rotation Trajectories

Eventually, in this section we give type I, II, and III minimum wheel-rotation trajectories up to symmetries. In Section V-A we described the symmetries of this problem. In the following we denote straight segment by $\mathbf{S}$, swinging around right and left wheels by $\mathbf{R}$ and $\mathbf{L}$ respectively, and rotation in place by $\mathbf{P}$. Directions are denoted by superscript $+$ and $-$ whenever it is required, otherwise it is constant throughout the trajectory. Forward and counter-clockwise are denoted by $+$, and backward and clockwise by $-$. Subscripts denote angles.

Lemma 3 implies that wheel-rotation is equal to the length of the curve that is traversed by the center of robot in the x-y plane along tight extremals. Since equations of motion of the differential-drive is the same as that of Reeds-Shepp car along a tight extremal, the center of robot in the x-y plane traverses a Reeds-Shepp curve along a tight minimum wheel-rotation trajectory. Here we use previous results about Reeds-Shepp curves in [11] to characterize tight minimum wheel-rotation trajectories.

Lemma 7. *If $\alpha > 0$ then $\mathbf{R}_\pi\mathbf{L}_\alpha$ is not minimum wheel-rotation.*

Proof. For any $\beta > 0$, we first show that $\mathbf{L}_\beta\mathbf{R}_\pi\mathbf{L}_\beta$ is not optimal. Observe that $\mathbf{L}^-_\beta\mathbf{R}^-_\pi\mathbf{L}^-_\beta$ has $(\pi+2\beta)b$ wheel rotation. Let $e = 4(1-\cos\beta)b$. The trajectory $\mathbf{R}^+_{\frac{\pi}{2}-\beta}\mathbf{S}^-_e\mathbf{R}^+_{\frac{\pi}{2}-\beta}$ has $(\pi-2\beta)b+e$ wheel rotation. Since $1-\cos\beta \leq \beta$ we must have $(\pi-2\beta)b+e \leq (\pi+2\beta)b$. Second, we show that $\mathbf{R}_\pi\mathbf{L}_\alpha$

TABLE I

MINIMUM WHEEL-ROTATION TRAJECTORIES SORTED BY SYMMETRY CLASS

	(A)	(B)	(C)	(D)	(E)	(F)
Base	$\mathbf{L}_\alpha^-\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{S}^+\mathbf{R}_\beta^-$	$\mathbf{L}_\alpha^-\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{S}^+\mathbf{L}_{\frac{\pi}{2}}^+\mathbf{R}_\beta^+$	$\mathbf{L}_\alpha^-\mathbf{R}_\gamma^-\mathbf{L}_\gamma^+\mathbf{R}_\beta^+$	$\mathbf{L}_\alpha^+\mathbf{R}_\gamma^-\mathbf{L}_\gamma^-\mathbf{R}_\beta^+$	$\mathbf{R}_\alpha^+\mathbf{P}_\gamma^+\mathbf{L}_\beta^+$	$\mathbf{P}_\alpha^+\mathbf{R}_\gamma^+\mathbf{P}_\beta^+$
$\mathcal{O}_1$	$\mathbf{R}_\beta^-\mathbf{S}^+\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{L}_\alpha^-$	$\mathbf{R}_\beta^+\mathbf{L}_{\frac{\pi}{2}}^+\mathbf{S}^+\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{L}_\alpha^-$	$\mathbf{R}_\beta^+\mathbf{L}_\gamma^+\mathbf{R}_\gamma^-\mathbf{L}_\alpha^-$	$\mathbf{R}_\beta^+\mathbf{L}_\gamma^-\mathbf{R}_\gamma^-\mathbf{L}_\alpha^+$	$\mathbf{L}_\beta^+\mathbf{P}_\gamma^+\mathbf{R}_\alpha^+$	$\mathbf{P}_\beta^+\mathbf{R}_\gamma^+\mathbf{P}_\alpha^+$
$\mathcal{O}_2$	$\mathbf{L}_\alpha^+\mathbf{R}_{\frac{\pi}{2}}^+\mathbf{S}^-\mathbf{R}_\beta^+$	$\mathbf{L}_\alpha^+\mathbf{R}_{\frac{\pi}{2}}^+\mathbf{S}^-\mathbf{L}_{\frac{\pi}{2}}^-\mathbf{R}_\beta^-$	$\mathbf{L}_\alpha^+\mathbf{R}_\gamma^+\mathbf{L}_\gamma^-\mathbf{R}_\beta^-$	$\mathbf{L}_\alpha^-\mathbf{R}_\gamma^+\mathbf{L}_\gamma^+\mathbf{R}_\beta^-$	$\mathbf{R}_\alpha^-\mathbf{P}_\gamma^-\mathbf{L}_\beta^-$	$\mathbf{P}_\alpha^-\mathbf{R}_\gamma^-\mathbf{P}_\beta^-$
$\mathcal{O}_3$	$\mathbf{R}_\alpha^+\mathbf{L}_{\frac{\pi}{2}}^+\mathbf{S}^+\mathbf{L}_\beta^+$	$\mathbf{R}_\alpha^+\mathbf{L}_{\frac{\pi}{2}}^+\mathbf{S}^+\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{L}_\beta^-$	$\mathbf{R}_\alpha^+\mathbf{L}_\gamma^+\mathbf{R}_\gamma^-\mathbf{L}_\beta^-$	$\mathbf{R}_\alpha^-\mathbf{L}_\gamma^+\mathbf{R}_\gamma^+\mathbf{L}_\beta^-$	$\mathbf{L}_\alpha^-\mathbf{P}_\gamma^-\mathbf{R}_\beta^-$	$\mathbf{P}_\alpha^-\mathbf{L}_\gamma^-\mathbf{P}_\beta^-$
$\mathcal{O}_1 \circ \mathcal{O}_2$	$\mathbf{R}_\beta^+\mathbf{S}^-\mathbf{R}_{\frac{\pi}{2}}^+\mathbf{L}_\alpha^+$	$\mathbf{R}_\beta^-\mathbf{L}_{\frac{\pi}{2}}^-\mathbf{S}^-\mathbf{R}_{\frac{\pi}{2}}^+\mathbf{L}_\alpha^+$	$\mathbf{R}_\beta^-\mathbf{L}_\gamma^-\mathbf{R}_\gamma^+\mathbf{L}_\alpha^+$	$\mathbf{R}_\beta^-\mathbf{L}_\gamma^+\mathbf{R}_\gamma^+\mathbf{L}_\alpha^-$	$\mathbf{L}_\beta^-\mathbf{P}_\gamma^-\mathbf{R}_\alpha^-$	$\mathbf{P}_\beta^-\mathbf{R}_\gamma^-\mathbf{P}_\alpha^-$
$\mathcal{O}_1 \circ \mathcal{O}_3$	$\mathbf{L}_\beta^+\mathbf{S}^+\mathbf{L}_{\frac{\pi}{2}}^+\mathbf{R}_\alpha^+$	$\mathbf{L}_\beta^-\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{S}^+\mathbf{L}_{\frac{\pi}{2}}^+\mathbf{R}_\alpha^+$	$\mathbf{L}_\beta^-\mathbf{R}_\gamma^-\mathbf{L}_\gamma^+\mathbf{R}_\alpha^+$	$\mathbf{L}_\beta^-\mathbf{R}_\gamma^+\mathbf{L}_\gamma^+\mathbf{R}_\alpha^-$	$\mathbf{R}_\beta^-\mathbf{P}_\gamma^-\mathbf{L}_\alpha^-$	$\mathbf{P}_\beta^-\mathbf{L}_\gamma^-\mathbf{P}_\alpha^-$
$\mathcal{O}_2 \circ \mathcal{O}_3$	$\mathbf{R}_\alpha^-\mathbf{L}_{\frac{\pi}{2}}^-\mathbf{S}^-\mathbf{L}_\beta^-$	$\mathbf{R}_\alpha^-\mathbf{L}_{\frac{\pi}{2}}^-\mathbf{S}^-\mathbf{R}_{\frac{\pi}{2}}^+\mathbf{L}_\beta^+$	$\mathbf{R}_\alpha^-\mathbf{L}_\gamma^-\mathbf{R}_\gamma^+\mathbf{L}_\beta^+$	$\mathbf{R}_\alpha^+\mathbf{L}_\gamma^-\mathbf{R}_\gamma^-\mathbf{L}_\beta^+$	$\mathbf{L}_\alpha^+\mathbf{P}_\gamma^+\mathbf{R}_\beta^+$	$\mathbf{P}_\alpha^+\mathbf{L}_\gamma^+\mathbf{P}_\beta^+$
$\mathcal{O}_1 \circ \mathcal{O}_2 \circ \mathcal{O}_3$	$\mathbf{L}_\beta^-\mathbf{S}^-\mathbf{L}_{\frac{\pi}{2}}^-\mathbf{R}_\alpha^-$	$\mathbf{L}_\beta^+\mathbf{R}_{\frac{\pi}{2}}^+\mathbf{S}^-\mathbf{L}_{\frac{\pi}{2}}^-\mathbf{R}_\alpha^-$	$\mathbf{L}_\beta^+\mathbf{R}_\gamma^+\mathbf{L}_\gamma^-\mathbf{R}_\alpha^-$	$\mathbf{L}_\beta^+\mathbf{R}_\gamma^-\mathbf{L}_\gamma^-\mathbf{R}_\alpha^+$	$\mathbf{R}_\beta^+\mathbf{P}_\gamma^+\mathbf{L}_\alpha^+$	$\mathbf{P}_\beta^+\mathbf{L}_\gamma^+\mathbf{P}_\alpha^+$
	$\alpha + \beta \leq \frac{\pi}{2}$	$\alpha + \beta \leq 2$	$\alpha, \beta < \gamma \leq \frac{\pi}{2}$	$\alpha, \beta < \gamma \leq \frac{\pi}{2}$	$\alpha + \gamma + \beta \leq \pi$	$\alpha + \gamma + \beta \leq \pi$

is not optimal. Let $0 < \epsilon < \alpha$ be a small positive number such that $2(1 - \cos\epsilon) < \epsilon$. We know that such ϵ exists. Let $g = 4(1 - \cos\epsilon)b$. Consider the trajectory $\mathbf{L}_\epsilon^+\mathbf{R}_{\frac{\pi}{2}-\epsilon}^+\mathbf{S}_g^-\mathbf{R}_{\frac{\pi}{2}-\epsilon}^+$ which has the same end configuration as $\mathbf{R}_\pi\mathbf{L}_\epsilon$. However, it has less wheel rotation than $\mathbf{R}_\pi\mathbf{L}_\epsilon$ because $g < 2b\epsilon$. Since any subpath of an optimal path should be optimal, $\mathbf{R}_\pi\mathbf{L}_\alpha$ is not optimal. ■

Theorem 2. *A type I minimum wheel-rotation trajectory is of the following forms:*

- $\mathbf{L}_\alpha^-\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{S}^+\mathbf{R}_\beta^-$
- $\mathbf{L}_\zeta^-\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{S}^+\mathbf{L}_{\frac{\pi}{2}}^+\mathbf{R}_\gamma^+$,

in which $\alpha + \beta \leq \frac{\pi}{2}$ *and* $\zeta + \gamma \leq 2$.

Proof. In Section V-B case 1, we showed that type I extremals are of the following forms:

- $(\mathbf{R}_\pi^-\mathbf{L}_\pi^-)^*\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{S}^+\mathbf{R}_{\frac{\pi}{2}}^-(\mathbf{L}_\pi^-\mathbf{R}_\pi^-)^*$
- $(\mathbf{R}_\pi^-\mathbf{L}_\pi^-)^*\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{S}^+\mathbf{L}_{\frac{\pi}{2}}^+(\mathbf{R}_\pi^+\mathbf{L}_\pi^+)^*$.

Lemma 7 shows that if $\eta > 0$ then $\mathbf{L}_\pi\mathbf{R}_\eta$ cannot be minimum wheel-rotation. It is enough to note that any subpath of an optimal path is necessarily optimal. Hence, the only possibilities are of the following form:

- $\mathbf{L}_\alpha^-\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{S}^+\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{L}_\eta^-$
- $\mathbf{L}_\zeta^-\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{S}^+\mathbf{L}_{\frac{\pi}{2}}^+\mathbf{R}_\gamma^+$,

in which $\alpha, \eta, \zeta, \gamma < \pi$. Assume $\alpha > 0$. We claim that $\eta = 0$, because a path of type $\mathbf{R}^+\mathbf{S}^-\mathbf{R}^+$ is shorter than $\mathbf{L}_\alpha^-\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{S}^+\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{L}_\eta^-$. Hence, $\mathbf{L}_\alpha^-\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{S}^+\mathbf{R}_\beta^-$ is possibly optimal in which $\beta \leq \frac{\pi}{2}$. If $\alpha > \frac{\pi}{2}$, then a path of type $\mathbf{R}^+\mathbf{L}_{\frac{\pi}{2}}^+\mathbf{S}^+\mathbf{R}^-$ is shorter than $\mathbf{L}_\alpha^-\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{S}^+$. Thus, $\alpha, \beta, \zeta, \gamma \leq \frac{\pi}{2}$. Also, charaterization of Reeds-Shepp curves of type $C|CSC$ in [11] implies that $\alpha + \beta \leq \frac{\pi}{2}$. Finally, if $\zeta + \gamma > 2$, then $\mathbf{L}_{\frac{\pi}{2}-\zeta}^+\mathbf{S}^-\mathbf{R}_{\frac{\pi}{2}-\gamma}^-$ is shorter than $\mathbf{L}_\zeta^-\mathbf{R}_{\frac{\pi}{2}}^-\mathbf{S}^+\mathbf{L}_{\frac{\pi}{2}}^+\mathbf{R}_\gamma^+$. Hence, $\zeta + \gamma \leq 2$. For such an optimal trajectory see (A) and (B) in Figure 1. ■

Theorem 3. *A type II minimum wheel-rotation trajectory is of the following forms:*

- $\mathbf{L}_\alpha^-\mathbf{R}_\gamma^-\mathbf{L}_\gamma^+\mathbf{R}_\beta^+$
- $\mathbf{L}_\alpha^+\mathbf{R}_\gamma^-\mathbf{L}_\gamma^-\mathbf{R}_\beta^+$,

in which $0 \leq \alpha, \beta < \gamma \leq \frac{\pi}{2}$.

Proof. In Section V-B case 3, we showed that type II extremals are of the form $(\mathbf{L}_\gamma^-\mathbf{R}_\gamma^-\mathbf{L}_\gamma^+\mathbf{R}_\gamma^+)^*$. We prove that two complete sets of four swings is not optimal, i.e. $\mathbf{R}_\gamma^+\mathbf{L}_\gamma^-\mathbf{R}_\gamma^-\mathbf{L}_\gamma^+\mathbf{R}_\gamma^+\mathbf{L}_\gamma^-\mathbf{R}_\gamma^-\mathbf{L}_\gamma^+$ is not optimal. In each set, the amount of robot displacement in x-y plane is $8b\sin^2\frac{\gamma}{2}$, in which γ is the angle of swings. If $0 < \gamma < \frac{\pi}{4}$, then let ζ be such that $\sin^2\frac{\zeta}{2} = 2\sin^2\frac{\gamma}{2}$. It follows that $\zeta < 2\gamma < \frac{\pi}{2}$. A type II extremal that is composed of four swings of angle ζ has less wheel rotation. If $\frac{\pi}{4} \leq \gamma \leq \frac{\pi}{2}$, then $b\pi + 16b\sin^2\frac{\gamma}{2} < 8b\gamma$, and the trivial trajectory which is composed of rotation in place, going straight, and again rotation in place gives less wheel rotation. A similar argument, based on what we just showed, proves that $\mathbf{L}_\gamma^-\mathbf{R}_\gamma^-\mathbf{L}_\gamma^+\mathbf{R}_\gamma^+\mathbf{L}_\gamma^-\mathbf{R}_\gamma^-\mathbf{L}_\gamma^+\mathbf{R}_\gamma^+$ is not minimum wheel-rotation either. Moreover, Lemma 3 implies that wheel-rotation is equal to the length of the curve that is traversed by the center of robot in the x-y plane along tight extremals. Since the center of robot in the x-y plane traverses a Reeds-Shepp curve along a tight minimum wheel-rotation trajectory, the only possibilities [11] are

- $\mathbf{L}_\alpha^-\mathbf{R}_\gamma^-\mathbf{L}_\gamma^+\mathbf{R}_\beta^+$
- $\mathbf{L}_\alpha^+\mathbf{R}_\gamma^-\mathbf{L}_\gamma^-\mathbf{R}_\beta^+$,

in which $\alpha, \beta < \gamma \leq \frac{\pi}{2}$. For such an optimal trajectory see (C) and (D) in Figure 1. ■

Lemma 8. *If* $\alpha > 0$ *then* $\mathbf{P}_{\pi-\gamma}\mathbf{R}_\gamma\mathbf{P}_\alpha$ *is not minimum wheel-rotation, in which* $0 \leq \gamma \leq \pi$.

Proof. It is enough to note that $\mathbf{P}_{\pi-\gamma}^-\mathbf{R}_\gamma^-\mathbf{P}_\alpha^-$ has $\pi + \alpha$ wheel rotation whereas $\mathbf{L}_\gamma^+\mathbf{P}_{\pi-\gamma-\alpha}^+$ has $\pi - \alpha$ wheel rotation. Since they connect the same initial and goal configurations, the former cannot be minimum wheel-rotation. ■

Lemma 9. *If* $0 \leq \zeta, \eta \leq \gamma \leq \pi$ *and* $\zeta + \eta > \gamma$ *then* $\mathbf{R}_\zeta\mathbf{P}_{\pi-\gamma}\mathbf{L}_\eta$ *is not minimum wheel-rotation.*

Proof. Suppose $\mathbf{R}_\zeta\mathbf{P}_{\pi-\gamma}\mathbf{L}_\eta$ is minimum wheel-rotation. Let $\delta = \gamma - \zeta$. By assumption we have $0 \leq \delta < \eta$. We replace the subpath $\mathbf{R}_\zeta^-\mathbf{P}_{\pi-\gamma}^-\mathbf{L}_\delta^-$ of $\mathbf{R}_\zeta^-\mathbf{P}_{\pi-\gamma}^-\mathbf{L}_\eta^-$ by an equivalent trajectory $\mathbf{L}_\delta^+\mathbf{P}_{\pi-\gamma}^+\mathbf{R}_\zeta^+$ to get $\mathbf{L}_\delta^+\mathbf{P}_{\pi-\gamma}^+\mathbf{R}_\zeta^+\mathbf{L}_{\eta-\delta}^-$. Boundary points and wheel rotation of this trajectory is equal to boundary points and wheel rotation of the original trajectory

$\mathbf{R}_\zeta^-\mathbf{P}_{\pi-\gamma}^-\mathbf{L}_\eta^-$. Hence, $\mathbf{L}_\delta^+\mathbf{P}_{\pi-\gamma}^+\mathbf{R}_\zeta^+\mathbf{L}_{\eta-\delta}^-$ is a minimum wheel-rotation trajectory. In particular, it must satisfy the PMP. This is a contradiction because $\mathbf{L}_\delta^+\mathbf{P}_{\pi-\gamma}^+\mathbf{R}_\zeta^+\mathbf{L}_{\eta-\delta}^-$ is not an extremal. ■

Theorem 4. *A type III minimum wheel-rotation trajectory is of the following forms:*

- $\mathbf{R}_\alpha\mathbf{P}_\gamma\mathbf{L}_\beta$
- $\mathbf{P}_\alpha\mathbf{R}_\gamma\mathbf{P}_\beta$,

in which $\alpha+\gamma+\beta\leq\pi$.

Proof. In Section V-C, we showed for any loose extremal there is an equivalent trajectory which is composed of swing and rotation in place, i.e. $(\mathbf{R}_\gamma\mathbf{P}_{\pi-\gamma}\mathbf{L}_\gamma\mathbf{P}_{\pi-\gamma})^*$. Lemma 8 implies that a 4-piece trajectory of this type cannot be minimum wheel-rotation. Thus, the only possible type III minimum wheel-rotation trajectories are of the following forms: $\mathbf{R}_\zeta\mathbf{P}_{\pi-\gamma}\mathbf{L}_\eta$ and $\mathbf{P}_\alpha\mathbf{R}_\gamma\mathbf{P}_\beta$, in which $\alpha,\beta\leq\pi-\gamma$ and $\zeta,\eta\leq\gamma$. If $\alpha+\gamma+\beta>\pi$ then $\mathbf{P}_\alpha^-\mathbf{R}_\gamma^-\mathbf{P}_\beta^-$ is not minimum wheel-rotation, because $\mathbf{P}_{\pi-\gamma-\alpha}^+\mathbf{R}_\gamma^+\mathbf{P}_{\pi-\gamma-\beta}^+$ is shorter. If $\zeta+\eta>\gamma$ then Lemma 9 proves that $\mathbf{R}_\zeta\mathbf{P}_{\pi-\gamma}\mathbf{L}_\eta$ is not minimum wheel-rotation. Hence, $\zeta+(\pi-\gamma)+\eta\leq\pi$, and by renaming parameters we obtain the result. For such an optimal trajectory see (E) and (F) in Figure 1. ■

Taking the symmetries in Section V-A into account, all the maximally optimal trajectories with their symmetric clones are given in Table I. Since the symmetry operators $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ commute, we do not need to worry about their order. Finally, we include the following lemma to compare minimum wheel-rotation with optimal time:

Lemma 10. *Let* $T^\star$ *be the optimal time given in [1] and* $J^\diamond$ *the minimum wheel-rotation. It follows that* $\frac{1}{2}T^\star\leq J^\diamond\leq T^\star$.

VII. Conclusions

We used the Filippov theorem to first prove that the minimum wheel-rotation trajectories exist for the differential drive. By applying the Pontryagin Maximum Principle [5] and developing geometric arguments, we derived optimality necessary conditions which helped to rule out non-optimal trajectories. The remaining trajectories form 28 different minimum wheel-rotation trajectories, which are listed in Table I. Based on the characterization of minimum wheel-rotation trajectories, it remains to further determine the applicable trajectory for every pair of initial and goal configurations.

As it was seen in Section VI, the cost of a tight minimum wheel-rotation trajectory is equal to the cost of an equivalent Reeds-Shepp curve. However, since loose minimum wheel-rotation trajectories are composed of rotation in place and swing segments, they are not identical with equivalent Reeds-Shepp curves. Regions of activity of different minimum wheel-rotation trajectories and the relationship between Reeds-Shepp cost and minimum wheel-rotation in general remain open questions.

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