Minimalism in Robotics: From Sensing to Filtering to Planning

PART 2: SENSING

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Physical Sensors

Physical state spaces
Sensor mapping
Basic Examples
Depth sensors
Detection sensors
Relational sensors
Gap sensors
Field sensors
Preimages
Sensor lattice
Additional complications
What Is a Sensor?

Physical Sensors

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Light-dependent resistor

GPS unit

Wireless card

Toilet float mechanism

We know it when we see it, but will not try to formally classify.
Where Might We Want to Use Sensors?

Physical Sensors
- Physical state spaces
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Shopping mall

Control room

Assisted living

Coral reef
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Spatial: displacement, velocity, acceleration, distance to something, proximity, position, attitude, area, volume, level/tilt, motion detection

Temporal: clock, chronometer (elapsed time), frequency.

Electromagnetic: voltage, current, power, charge, capacitance, inductance, magnetic field, light intensity, color. These may operate within a circuit or within open space.

Mechanical: solid (mass, weight, density, force, strain, torque), fluid (acoustic, pressure, flow, viscosity), thermal (temperature), calories.

Other: chemical (composition, pH, humidity, pollution, ozone), radiation (nuclear), biomedical (blood flow, pressure).

See CRC Measurement, Instrumentation, and Sensors Handbook
What Sensors Are Available?

Physical Sensors
- Physical state spaces
- Sensor mapping

Basic Examples
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Contact sensor
- Sonar
- Compass
- Microphone
What Sensors Are Available?

**Physical Sensors**

- Physical state spaces
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**Wheel encoder**

**Stopwatch/timer**

**Occupancy detector**

**Safety beam**
What Sensors Are Available?

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Camera

Wii remote

Pressure mat

SICK laser scanner
Common Sensor Characteristics

- **Transfer function** converts physical phenomenon to sensor reading:
  \[ g : \mathbb{R} \rightarrow \mathbb{R}. \]

- Domain of \( g \) may be **absolute** vs. **relative**.
- \( g \) itself may be **linear** or **nonlinear**.
- **Resolution** is given by set of possible \( g(x) \).

- **Sensitivity** is set of stimuli that produce same reading.
- **Repeatability** is producing same readings under same phenomena.
- **Calibration** eliminates systematic errors.

You will find these notions in sensor handbooks.
Physical state spaces
Physical Sensors vs. Virtual Sensors

**Physical sensor:** The real thing.

![Physical sensor image](image)

**Virtual sensor:** Mathematical model of information obtained from a sensing system.

A virtual sensor could have many alternative physical-sensor implementations.

Identifying which *virtual* sensor is required will lead to better filter design and planning algorithms.
The key idea in this section is to understand how two spaces are related:

1. The *physical state space*, in which each physical state is a cartoon-like description of the possible world external to the sensor.
2. The *observation space*, which is the set of possible sensor output values or observations.

Physical state $\rightarrow$ a sensor observation
Observation: The wall is 3 meters away.
What possible external physical worlds are consistent with that?
A Common Structure

- Localization only: Set of possible configurations
- Mapping only: Set of possible environments
- Both: Set of configuration-environment pairs

Let $\mathcal{Z}$ be any set of sets.

Each $Z \in \mathcal{Z}$ is a “map”.
Each $z \in Z$ is the configuration or “place” in the map.

Unknown configuration and map yields a state space as:
All $(z, Z)$ such that $z \in Z$ and $Z \in \mathcal{Z}$.  

Without any obstacles:

- Any position \((q_x, q_y) \in \mathbb{R}^2\)
- Any orientation \(q_\theta \in [0, 2\pi)\)
- Let state space \(X\) be all positions and orientations

Can imagine \(X \subset \mathbb{R}^3\); however, for orientation, we have additional topology since \(q_\theta = 0 = 2\pi\).

Could write \(X = \mathbb{R}^2 \times S^1\), in which \(S^1\) is a circle and the set of all orientations.

Could write \(X = SE(2)\), set of all 2D rigid-body transformations.
Suppose $E \subset \mathbb{R}^2$ is known to be the set of allowable positions.

Must have $(q_x, q_y) \in E$.

State space: $X = E \times S^1$
Given a set of $k$ possible maps:

$$\mathcal{E} = \{E_1, E_2, \ldots, E_k\}$$

For example, could be given 5 maps:

$$\mathcal{E} = \{E_1, E_2, E_3, E_4, E_5\}$$

$X$ is all $(q, E_i)$ in which $(q_x, q_y) \in E_i$ and $E_i \in \mathcal{E}$.

Recall the common structure.
Given an infinite map family, $\mathcal{E}$, of environments.

Examples:

- The set of all connected, bounded polygonal subsets that have no interior holes (formally, they are *simply connected*).
- The previous set expanded to include all cases in which the polygonal region has a finite number of polygonal holes.
- All subsets of $\mathbb{R}^2$ that have a finite number of points removed.
- All subsets of $\mathbb{R}^2$ that can be obtained by removing a finite collection of nonoverlapping discs.
- All subsets of $\mathbb{R}^2$ obtained by removing a finite collection of nonoverlapping convex sets.
- A collection of piecewise-analytic subsets of $\mathbb{R}^2$. 
In spite of larger $\mathcal{E}$, there is no difference:

$X$ is all pairs $(q, E)$ in which $(q_x, q_y) \in E$ and $E \in \mathcal{E}$.

We can write $X \subset \mathbb{R}^2 \times S^1 \times \mathcal{E}$.

$X$ is enormous! But that is fine here. We do not compute directly on it.

Note: Putting useful probability densities over $X$ might be difficult or impossible.

$X$ is usually **not a manifold** (doesn’t look like C-space)
Place a *body* $B$ into $E$.

Each could have a configuration space $SE(2)$, so that we transform it: $B(q_x, q_y, q_\theta) \subset E$.

Here, assume every body is a point, except for obstacles. Otherwise, see Chapter 4 of *Planning Algorithms* for configuration space obstacles.
Terms for Bodies

- **Robot**: A body that carries sensors, performs computations, and executes motion commands.
- **Landmark**: Usually a small body that has a known location and is easily detectable and distinguishable from others.
- **Object**: A body that can be detected and manipulated by a robot. It can be carried by a robot or dropped at a location.
- **Pebble**: A small object that is used as a marker to detect when a place has been revisited.
- **Target**: A person, a robot, or any other moving body that we would like to monitor using a sensor.
- **Obstacle**: A fixed or moving body that obstructs the motions of others.
- **Evader**: An unpredictable moving body that attempts to elude detection.
- **Treasure**: Usually a stationary body that has an unknown location but is easy to recognize by a sensor directly over it.
- **Tower**: A body that transmits a signal, such as a cell-phone tower or a lighthouse.
Names of bodies are not important.

Instead, the properties that affect mathematical models are crucial:

1. What are its *motion capabilities*?
2. Can it be *distinguished* from other bodies?
3. How does it *interact* with other bodies?
Body Property 1: Motion Capabilities

There are three possibilities:

1. **If static**, then it never moves.
   
   *Examples: Most landmarks, obstacles, a tower*

2. It may have **predictable** motion.
   
   *Examples: A rolling ball, a pendulum, a robot*

3. It may have **unpredictable** motion.
   
   *Examples: An evader, a target*

If planning is involved, then another issue is whether or not the body can be commanded to move.
Take any collection of distinct bodies $B_1, \ldots, B_n$.

Let $\sim$ be any equivalence relation:

$B_i \sim B_j$ if and only if they cannot be distinguished from each other.

Example: Could assign *labels* to be bodies. With humans, we have *women* and *men*.

Warning: Sometimes indistinguishability might not be transitive!
Three interaction types are generally possible between a pair $B_1, B_2$, of bodies:

- **Sensor obstruction:** Suppose a sensor would like to observe information about body $B_1$. Does body $B_2$ interfere with the observation?

- **Motion obstruction:** Does body $B_2$ obstruct the possible motions of body $B_1$? If so, then $B_2$ becomes an obstacle that must be avoided.

- **Manipulation:** In this case, body $B_1$ could cause body $B_2$ to move. For example, if $B_2$ is an obstacle, then $B_1$ might push it out of the way.
A field is a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, with $n = 2$ or $n = 3$ and $m \leq n$.

**Examples:**

- Encoding $E$: $f : \mathbb{R}^2 \rightarrow \{0, 1\}$ in which $f(q_x, q_y) = 1$ if and only if $(q_x, q_y) \in E$.
- An altitude map: $f : \mathbb{R}^2 \rightarrow [0, \infty)$.
- An intensity field: $f : \mathbb{R}^2 \rightarrow [0, \infty)$.
- An electromagnetic field: $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. 
Of course the world is not static.

Let $T$ be *time interval*; usually, $T = [0, \infty)$.

Using any state space $X$, define *state-time space*:

$$Z = X \times T$$

Each $z \in Z$ is a pair $z = (x, t)$ and $x$ is the state at time $t$.

No, not this:
A state trajectory $\tilde{x}$ is a time-parameterized path through $X$:

$$\tilde{x} : T \rightarrow X$$

Sometimes, domain of $\tilde{x}$ may be only $[0, t]$.

Could take time derivatives of states and expand state space. We will not do that here.
Sensor mapping

Basic Examples

Depth sensors

Detection sensors

Relational sensors

Gap sensors

Field sensors

Preimages

Sensor lattice

Additional complications
Let $X$ be any physical state space.

Let $Y$ denote the **observation space**, which is the set of all possible sensor observations.

A virtual sensor is defined by a **sensor mapping**:

$$ h : X \rightarrow Y. $$

Note similarity to transfer function for physical sensors.

When $x \in X$, the sensor instantaneously observes $y = h(x) \in Y$. 

Using the sensor mapping, we will make many models:

- Basic (boring) examples
- Depth sensors
- Detection sensors
- Relational sensors
- Gap sensors
- Field sensors

Purpose: To define models of *information* to be used in filters.

Remember: Virtual sensors could have many physical implementations.
Basic Examples
Basic Examples: The Two Extremes

The weakest possible sensor

**Dummy Sensor:**
\[ Y = \{0\} \] and \[ h(x) = 0 \] for all \[ x \in X \]

The strongest possible sensor(s)

**Identity Sensor:**
\[ Y = X \] and \[ y = h(x) = x \]
Just give me the state!

**Bijective Sensor:**
\[ h \] is bijective function from \( X \) to \( Y \).
\( x \) can be reconstructed as \( x = h^{-1}(y) \).
Basic Examples: Linear Sensors

\[ X = Y = \mathbb{R}^3 \]

**LINEAR SENSOR:**

Let \( y = h(x) = Cx \) for 3 by 3 matrix \( C \).

If \( C \) has full rank, then \( h \) is a bijective sensor.

If \( C \) has lower rank, then lines or planes produce same observation.

Linear sensors used widely in control theory.
**PROJECTION SENSOR:**

Choose some components of $X$.

$X = \mathbb{R}^3$ and $x = (x_1, x_2, x_3) \in X$.

$Y = \mathbb{R}^2$

$y = h(x) = (x_1, x_2)$

$X = \mathbb{R}^2 \times S^1$

A state is $(q_x, q_y, q_\theta) \in X$.

Position sensor: Observes $(q_x, q_y)$ and leaves $q_\theta$ unknown.

Ideal compass: Observes $q_\theta$ and leaves $q_x$ and $q_y$ unknown.
Depth sensors
Observe the distance to the boundary of $E$.

State space: $X \subset SE(2) \times \mathcal{E}$
State: $x = (q_x, q_y, \theta, E)$ with $(q_x, q_y) \in E$ and $E \in \mathcal{E}$. 
**Directional Depth Sensor**

**Directional depth sensor:**

\[ h_d(p, \theta, E) = \|p - b(x)\| \]

Let \( p = (q_x, q_y) \) and \( \theta = q_\theta \) (shorthand notation)

\( b(x) \) is point on boundary \( \partial E \) hit by ray.
Boundary Distance Sensor

**Boundary distance sensor:**

\[ h_{bd}(p, \theta, E) = \min_{\theta' \in [0, 2\pi)} h_d(p, \theta', E) \]

No dependency on \( \theta \)
Fix some $\epsilon > 0$.

**PROXIMITY SENSOR:**

$$h_{p\epsilon}(p, \theta, E) = \begin{cases} 
1 & \text{if } h_{bd}(p, \theta, E) \leq \epsilon \\
0 & \text{otherwise} 
\end{cases}$$

Detects whether within $\epsilon$ of the boundary.

**BOUNDARY SENSOR:**

$$h_{bd}(p, \theta, E) = \begin{cases} 
1 & \text{if } h_{bd}(p, \theta, E) = 0 \\
0 & \text{otherwise} 
\end{cases}$$

Detects whether boundary is contacted.
**SHIFTED DIRECTIONAL DEPTH SENSOR:**
Robot oriented along $\theta$, but sensor is offset by $\phi$

$$h_{sd\phi}(p, \theta, E) = \|p - b(p, \theta + \phi, E)\|$$
**K-DIRECTIONAL DEPTH SENSOR:**

Let $k$ be number of directions.

The observation is a vector $y = (y_1, \ldots, y_k)$

$$y_i = h_i(p, \theta, E) = h_{sd\phi_i}(p, \theta, E).$$
Like an infinite-dimensional vector of observations

**OMNIDIRECTIONAL DEPTH SENSOR:**
\[ h_{od}(x) = y, \text{ in which } y : S^1 \rightarrow [0, \infty) \]

\[ y(\phi) = h_{od\phi}(p, \theta, E). \]
Omnidirectional Depth Sensor

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How does the observation $y : S^1 \to [0, \infty)$ look?

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How does the observation $y : S^1 \rightarrow [0, \infty)$ look?
Depth Sensors: Practical Limits

Limited angle:
\[ y : [\phi_{min}, \phi_{max}] \rightarrow [0, \infty) \]

Limited depth:
\[
h_{dd}(p, \theta, E) = \begin{cases} 
  d(x) & \text{if } d_{min} \leq d(x) \leq d_{max} \\
  \# & \text{otherwise}
\end{cases}
\]
Detection sensors
New category: Detection Sensors

Is a body in the field of view, or *detection region*?
Detection Sensors: Fundamental Aspects

Three fundamental questions:

1. Can the sensor move? For example, it could be mounted on a robot or it could be fixed to a wall.
2. Are the bodies so large relative to the range of the sensor that the body models cannot be simplified to points?
3. Can the sensor provide additional information that helps to classify a body within its detection region?

Simplest case: Answer “no” to all three questions.
This is the simplest case.

**Static Binary Detector:**

\[ h(p, E) = \begin{cases} 
1 & \text{if } p \in V \\
0 & \text{otherwise} 
\end{cases} \]

Simply indicates whether the body is in \( V \).
Moving Binary Detector

$q$ is configuration of the body carrying the sensor.

$V(q)$ is the configuration-dependent detection region.

**Moving Binary Detector:**

$$h(p, E) = \begin{cases} 
1 & \text{if } p \in V(q) \\
0 & \text{otherwise}
\end{cases}$$

$V$ has simply been replaced by $V(q)$.
A body has configuration \( q' \) and \( B(q') \subset E \).

**DETECTING LARGER BODIES:**

\[
h(q', E) = \begin{cases} 
1 & \text{if } B(q') \cap V \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]

Looks like obstacle regions in configuration space!
There are $n$ points bodies, $P = \{p_1, \ldots, p_n\}$.

**State:** $x = (q, p_1, \ldots, p_n, E)$, in which $q$ is sensor configuration.

**At-Least-One-Body Detector:**

$$h(q, p_1, \ldots, p_n, E) = \begin{cases} 
1 & \text{if for any } i, p_i \in V(q) \\
0 & \text{otherwise}
\end{cases}$$

Sensor detects when at least one of the bodies is in $V(q)$. 
**Body Counter:**

\[
h(q, p_1, \ldots, p_n, E) = |P \cap V(q)|
\]

If number of bodies generally unknown, but sensors fixed and environment \(E\) known:

\[
X = \{\#\} \cup E \cup E^2 \cup E^3 \cup E^4 \ldots
\]
$L$ is a set of class labels.

$\ell$ is an assignment mapping:

$$\ell : \{1, \ldots, n\} \rightarrow L$$

**Labeled-Body Detector:**

$$h_\lambda(p, E) = \begin{cases} 
1 & \text{if for some } i, p_i \in V \text{ and } \ell(i) = \lambda \\
0 & \text{otherwise}
\end{cases}$$

Examples: Each body is a man, dog, tree, car, ...
Relational sensors
Consider any relation $R$ on the set of all bodies.

For a pair of bodies, $B_1$ and $B_2$, examples of $R(B_1, B_2)$ are:

- $B_1$ is in front of $B_2$
- $B_1$ is to the left of $B_2$
- $B_1$ is on top of $B_2$
- $B_1$ is closer than $B_2$
- $B_1$ is bigger than $B_2$.

More precisely, Let $R_x(i, j)$ mean $B_i$ is related to $B_j$, when the system is at state $x$.

Idea is due to Guibas
**PRIMITIVE RELATIONAL SENSOR**: 

\[
h(x) = \begin{cases} 
1 & \text{if } R_x(i, j) \\
0 & \text{otherwise} 
\end{cases}
\]

Simply detects whether the relation is satisfied for bodies \(B_i\) and \(B_j\).

Using this, we can form *compound relational sensors*.
Relation: “is to the left of”

Observation: \( y = (4, 2, 1, 3, 5) \)
Observation space: \( Y \) is all \( 5! \) permutations.
Relation: “is closer than”

Observation: $y = (2, 3, 5, 4, 1)$
Relation: “is to the left of in counterclockwise order”

Observation: $y = (1, 2, 4, 3, 5)$

Note that $y$ could equivalently be $(4, 3, 5, 1, 2)$. 
Gap sensors
Report information obtained along the boundary of $V(q)$, which is denoted as $\partial V(q)$

Two qualitatively different parts of $\partial V(q)$:

1. A piece of a body boundary
2. A gap (discontinuity in depth)

A gap sensor reports how these parts alternate.
**Simple Gap Sensor**

**Simple Gap Sensor:**

Alternating between boundary and gaps:

\[ y = (B_0, g_1, B_0, g_2, B_0, g_3, B_0, g_4, B_0, g_5) \]

Equivalently:

\[ y = (g_1, g_2, g_3, g_4, g_5) \]
A new kind of gap, due to being out of range: $G_i$

**Depth-limited gap sensor:**

$y = (B_0, G_1, B_0, g_1, G_2, g_2, B_0, g_3, G_3, g_4)$
**Multibody Gap Sensor**

**MULIBODY GAP SENSOR:**

\[ y = (G_1, g_1, B_4, g_2, B_5, g_3, B_4, g_4, G_2, g_5, B_3, g_6, B_2, g_7, B_1) \]
**Landmark Counter**

Landmark Counter:

\[ y = (3, 3, 4, 0, 1) \]

Equivalent to combinatorial visibility vector from Gfeller et al. 2007.
Field sensors
Direct field sensor:

\[ h(x) = h(p, \theta) = (f_1(p), f_2(p)) \]

Vectors appear with respect to global frame orientation.
Field Sensors: Scalars

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**DIRECT INTENSITY SENSOR:**

\[ h(x) = h(p, \theta) = \| f(p) \| \]

**INTENSITY ALARM:**

\[
\begin{align*}
  h(p, \theta) &= \begin{cases} 
    1 & \text{if } \| f(p) \| \geq \epsilon \\
    0 & \text{otherwise}
  \end{cases}
\end{align*}
\]
Unknown monotonically increasing function:

\[ g : [0, \infty) \rightarrow [0, \infty) \]

**TRANSFORMED INTENSITY:**

\[ h(x) = g(\| f(x) \|) \]
More realistically, vectors observed in local orientation frame.

$R(\phi)$ is $2 \times 2$ rotation matrix by $\phi$.

field vector observation:

$$h_{fv}(x) = R(-\theta)f(p)$$

If $f$ is given and $\theta$ is unknown, then it can be determined using $h_{fv}(x)$.

Likewise, if $\theta$ is known and $f$ is unknown, then $f(p)$ can be determined from $f(p) = R(\theta)h_{fv}(x)$.
Field Direction Observation

Let $y' = h_{f_v}(x)$.

**FIELD DIRECTION OBSERVATION:**

$$y = h_{fdo}(x) = \text{atan2}(y'_2, y'_1)$$

Special case: An ideal magnetic compass, $f(p) = (0, 1)$.

The orientation $\theta$ can be recovered from the given field.
Preimages
The amount of state uncertainty due to a sensor

\[ h : X \rightarrow Y \]

The preimage of an observation \( y \) is

\[ h^{-1}(y) = \{ x \in X \mid y = h(x) \} \]

Think about the uncertainty being handled here!
Suppose $X$ and $h : X \rightarrow Y$ are given.

The set of all preimages partitions $X$.

There is one preimage for every $y \in Y$.

Let $\Pi(h)$ be the partition $X$ that is induced by $h$. 
- \( n \) point bodies move in \( \mathbb{R}^2 \).
- \( X = \mathbb{R}^{2n} \)
- \( Y = \{0, 1, \ldots, n\} \)
- The sensor mapping \( h : X \rightarrow Y \) counts how many points lie in a fixed detection region \( V \).

For \( n = 4 \), there are 5 equivalence classes in \( \Pi(h) \).
Recall directional depth sensor.

For a known environment, $X = E \times S^1$.

The preimages are chunks of $SE(2)$.

What happens for an unknown environment?

The preimages are chunks of $\mathbb{R}^2 \times S^1 \times \mathcal{E}$. 
Sensor lattice
Comparing the Power of Sensors

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Question: Is better than ?

It is better to compare virtual sensors...

Fix the state space $X$.

Take any two sensors, $h_1 : X \rightarrow Y_1$ and $h_2 : X \rightarrow Y_2$.

$h_1$ dominates $h_2$ if and only if $\Pi(h_1)$ is a refinement of $\Pi(h_2)$.

This is denoted as $h_1 \succeq h_2$. 
If \( h_1 \succeq h_2 \), then \( h_2 \) can be “simulated” using only observations from \( h_1 \).

If \( \Pi(h_1) \) is a refinement of \( \Pi(h_2) \), then we can figure out what observation \( h_2 \) must make, using only \( y_1 \).

This is interpreted as the existence of a function \( g : Y_1 \to Y_2 \).

\[
\begin{array}{c}
X \\
\downarrow h_1 \\
\downarrow h_2 \\
Y_1 \\
\downarrow g \\
Y_2
\end{array}
\]

What about computability or complexity of \( g \)?
Comparing More Sensors

Fix the state space $X$

We could have a sensor chain:

$$h_1 \succeq h_2 \succeq h_3 \succeq h_4 \succeq h_5$$
Comparing More Sensors

Fix the state space $X$

We could have a sensor chain:

$$h_1 \preceq h_2 \preceq h_3 \preceq h_4 \preceq h_5$$

We could have a sensor tree:

Could we even have a directed acyclic graph?
For any set $X$, the set of all partitions forms a complete lattice.

\[ X = \{1, 2, 3, 4\} \]

Every pair has a glb and lub.
Fix $X$ and consider the set of all possible sensors $h : X \rightarrow Y$.

Above, $Y$ is not fixed!!

We say two sensors $h_1$ and $h_2$ are equivalent if and only if

$$\Pi(h_1) = \Pi(h_2).$$

Really, the partition of $X$ is the sensor model.

The set of all partitions of $X$ forms the sensor lattice.

All sensor models embed into this lattice!

The bijective sensor and dummy sensor are at the top and bottom, respectively.
Additional complications
Nondeterministic sensor mapping:

\[ h : X \rightarrow \text{pow}(Y) \]

Corresponding preimage definition:

\[ h^{-1}(y) = \{ x \in X \mid y \in h(x) \} \]

A sensor mapping induces a cover \( C(h) \) of \( X \), instead of a partition.
One-Dimensional Position Sensor

One-dimensional position sensor:

\[ h(x) = \{ y \in Y \mid |x - y| \leq \epsilon \} \]

For example, \( h(2) = [2 - \epsilon, 2 + \epsilon] \)

The preimage of an observation \( y \) is

\[ h^{-1}(y) = \{ x \in X \mid |x - y| \leq \epsilon \}. \]

Clearly, a cover of \( X = \mathbb{R} \) is induced by \( h \).
Two kinds of mistakes:

1. **False positive**: \( h(p, E) = 1 \) even though \( p \notin V \)
2. **False negative**: \( h(p, E) = 0 \) even though \( p \in V \)

What does \( C(h) \) look like?

\( X \) is completely covered by two preimages.
Fix accuracy, $\epsilon \geq 0$.

Inaccurate Directional Depth Sensor:

$$h_\epsilon(p, \theta, E) = \{ y \in [0, \infty) \mid ||p - b(x) - y|| \leq \epsilon \}.$$
The sensor mapping is replaced by:

\[ p(y|x) \]

Using nondeterministic \( h \) that we can declare \( p(y|x) = 0 \) for all \( y \notin h(x) \).
Error model: Gaussian with zero mean and variance $\sigma^2$

**Probabilistic 1D Position Sensor:**

$$p(y|x) = \frac{1}{\sqrt{\Sigma} (2\pi)^{k/2}} e^{(y-x)^T \Sigma^{-1} (y-x)}.$$
Error model: Gaussian with zero mean and $\Sigma$ as a $k \times k$ covariance matrix.

**Probabilistic General Position Sensor:**

$$p(y|x) = \frac{1}{|\Sigma|^{1/2}(2\pi)^{k/2}} e^{(y-x)^T \Sigma^{-1} (y-x)}.$$
Now probabilities are assigned to false positives and false negatives.

False positive probability:
\[ p(y = 1 \mid p \notin V) \]

False negative probability:
\[ p(y = 0 \mid p \in V) \]

If these probabilities are small, then the sensor is quite informative.
Again assume zero-mean Gaussian error density.

**PROBABILISTIC DIRECTIONAL DEPTH SENSOR:**

\[
p(y|p, \theta, E) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\|p-b(x)\|)^2}{2\sigma^2}}
\]
Recall state-time space, $Z = X \times T$.

Sensor mapping:

$$h : Z \rightarrow Y$$

$$y = h(z), \text{ or equivalently, } y = h(x, t)$$

Consider preimages, partitions of $Z$, and sensor lattice.

$$h^{-1}(y) = \{(x, t) \in Z \mid y = h(x, t)\}$$
**Perfect clock model:**

\[ y = h(z) = h(x, t) = t. \]
**Detector with Time Stamp:**

\[
h(p, E, t) = \begin{cases} 
(1, t) & \text{if } p \in V \text{ at time } t \\
(0, t) & \text{otherwise}
\end{cases}
\]
State trajectory: $\tilde{x} : [0, t] \to X$

Let $\tilde{X}$ be set of all state trajectories.

History-based sensor mapping:

$$ h : \tilde{X} \to Y $$

Preimages again:

$$ h^{-1}(y) = \{ \tilde{x} \in \tilde{X} \mid y = h(\tilde{x}) \} $$

$h$ induces a partition of $\tilde{X}$.

A history-based sensor lattice is obtained over $\tilde{X}$. 
Linear Odometer

Linear velocity of planar robot: \((v_x, v_y)\)

**LINEAR ODOMETER:**

\[
y = \theta_0 + \int_0^t \sqrt{v_x^2 + v_y^2} \, ds
\]

\(v_x\) and \(v_y\) are part of the state.

For example, \(x = (p_x, p_y, \theta, v_x, v_y)\)
Angular Odometer:

\[ y = \theta_0 + \int_0^t \dot{\theta}(s) ds \]
Observe what the state was one second ago.

Delayed measurement sensor:

\[ y = \begin{cases} 
\tilde{x}(t - 1) & \text{if } t \geq 1 \\
\# & \text{otherwise}
\end{cases} \]

\# means no measurement yet available.
Fixed time increment $\Delta t > 0$.

$\tilde{p}(t)$ is robot position in $\mathbb{R}^2$ at time $t$.

**Discrete-time odometer:**

$$h(\tilde{x}) = \sum_{i=1}^{\lfloor t/\Delta t \rfloor} \| \tilde{p}(i\Delta t) - \tilde{p}((i - 1)\Delta t) \|$$

This yields an estimate of the total distance travel.

It looks like a temporal filter, which is coming soon.
Physical sensors and their characteristics

Virtual sensors vs. physical sensors

Families: Depth, detection, relational, gap, field

Uncertainty comes from preimages!

The sensor lattice

Disturbances, history-based, state-time

To make better filters and planners, you need to find the appropriate virtual sensors for your task.