# Bang-Bang Boosting of RRTs

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***Abstract*— This paper explores the use of time-optimal controls to improve the performance of sampling-based kinodynamic planners. A computationally efficient steering method is introduced that produces time-optimal trajectories between any states for a vector of double integrators. This method is applied in three ways: 1) to generate RRT edges that quickly solve the two-point boundary-value problems, 2) to produce an RRT (quasi)metric for more accurate Voronoi bias, and 3) to time-optimize a given collision-free trajectory. Experiments are performed for state spaces with up to 2000 dimensions, resulting in improved computed trajectories and orders of magnitude computation time improvements over using ordinary metrics and constant controls.**

## I. INTRODUCTION

Rapidly exploring random trees were originally introduced as an approach to motion planning with differential constraints and dynamics [22]. The idea was to incrementally grow a space-filling tree by applying controls so that two-point boundary-value problems could be avoided if popular methods such as probabilistic roadmaps [19] were applied to these problems. Curiously, RRTs have found more success over the past decades for basic path planning (no differential constraints and dynamics), rather than their intended target, the *kinodynamic planning problem* [7].

Although this phenomenon is partly due to larger mainstream interest since the 1990s in basic path planning, it is primarily due to the additional challenges posed by the harder problems. Computational performance depends greatly how well the RRT nearest-neighbor metric approximates the true, optimal cost-to-go function, which is presumably unattainable. Furthermore, efficient steering methods or motion primitives are often needed to enhance performance, rather than applying constant controls as in [22]. Indeed, the most successful kinodynamic RRT planning methods have exploited the existence of simple cost-to-go functions and steering methods (ignoring obstacles) for special classes of systems [9], [11], [28], [31], [33] or have relied on numerical solutions computed offline for more general systems and optimization objectives [30]. Learning-based approaches to estimate the cost-to-go function have also been proposed [4], [23], [27], [34]. Inspired by all of these works, we enhance RRT performance by using metrics and steering methods based on bang-bang time-optimal controls [29]. It was already shown that RRT exploration seems to improve with bang-bang metrics [11], [17].

We continue in this direction by developing simple metrics and steering methods based on algebraic analysis which reduces steering to piecewise-constant controls, and trajectories to parabolic arcs in the configuration and state (phase) spaces. We then show experimentally that RRT performance improves by about orders of magnitude when compared to the original method that uses weighted Euclidean metrics and constant controls over fixed time intervals. The study presented here is focused on double integrator dynamics, which lies at the core of fully actuated systems, with the intention of extending it to more general dynamics of arbitrary stabilizable systems (similar to the way it was accomplished in [15]).

This paper also presents methods that rapidly optimize collision-free trajectories by iteratively applying the simple time-optimal steering method to the output of sampling-based planners. We consider two cases: 1) directly optimizing the result of a kinodyamic RRT-based planners, and 2) converting the piecewise-linear path produced by an RRT-based planner for basic path planning [20] into a trajectory by applying bang-bang controls along each segment and then further iteratively optimizing the result. Our experiments indicate that the second method is more efficient; however, it is limited to problems in which the initial and goal states are at zero velocity (if some form of completeness is demanded). An alternative to these optimizations would be to apply asymptotically optimal extensions of RRTs, such as RRT* [18] or SST* [24]; however, we are motivated by the evidence that "plan first and optimize later" often produces optimal paths more quickly and consistently than asymptotically optimal planning [14], [26].

The paper continues as follows. Section II defines the problem. Section III develops the algebraic details of bang-bang time optimal control for single and multiple, parallel double integrators. Section IV presents the new planning and optimization methods. Section V gives implementation details, computed results, and performance analysis. Section VI summarizes the results, their implications, and discusses the logical next steps.

## II. PROBLEM DEFINITION

Let $\mathcal{C}$ be the robot *configuration space*, assumed to be an $n$-dimensional smooth manifold with points referenced using local coordinates on $\mathbb{R}^n$. Geometric models (typically piecewise linear) are given for the robot and its (static) environment, and the robot model transforms for each $q$ depending on robot kinematics. Let $\mathcal{C}_{free}$ be the open subset of $\mathcal{C}$ in which the robot does not intersect obstacles. See [5], [21] for more details.

The first problem is:

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**Problem 1 (Basic path planning)** *Given any* $q_I, q_G \in \mathcal{C}_{free}$, *compute a path* $\tau : [0,1] \to \mathcal{C}_{free}$ *such that* $\tau(0) = q_I$ *and* $\tau(1) = q_G$.

This problem ignores kinematic constraints and dynamics, leading to a second, harder problem that takes these into account. Building upon $\mathcal{C}$ and $\mathcal{C}_{free}$, let $x = (q, \dot{q})$ be a $2n$-dimensional *state* vector for every configuration $q \in \mathcal{C}$. The set of all $x$ forms $X$, the *state space*, which is the tangent bundle $T(\mathcal{C})$. Assume $\mathcal{C}_{free}$ is lifted into $X$ as $X_{free} = \{(q, \dot{q}) \in X \mid q \in \mathcal{C}_{free}\}$.

Let $\dot{x} = f(x, u)$ define a standard *control system* on $X$, in which $u$ belongs to a compact *action set* $U \subset \mathbb{R}^m$. For convenience in this paper, we will equivalently define the control system in terms of the accelerations that actions $u \in U$ induce at a particular $q \in \mathcal{C}$. Thus, let $A(x) = A(q, \dot{q})$ be the set of all accelerations $\ddot{q}$ that can be obtained at $(q, \dot{q})$ by applying an action $u \in U$ for the system $f$. Let $a \in A(x)$ denote a particular acceleration vector ($a = \ddot{q}$) that may be applied. Let $A = \cup_{x \in X} A(x)$. Let $\Phi(x, \tilde{a})$ denote the state trajectory $\tilde{x} : [0, t_F] \to X$ obtained by integrating a control $\tilde{a} : [0, t_F] \to A$ from state $x$. This leads to the next problem:

**Problem 2 (Kinodynamic planning)** *Given any* $x_I, x_G \in X_{free}$, *compute an acceleration* control $\tilde{a} : [0, t_F] \to A$ *for which* $a(t) \in A(\tilde{x}(t))$ *for all* $t \in [0, t_F]$, *and that produces a state trajectory* $\tilde{x} = \Phi(x_I, \tilde{a})$, *with* $\tilde{x} : [0, t_F] \to X_{free}$ *and* $\tilde{x}(t_F) = x_G$.

These restricted versions will be considered in this paper:

1) *Stabilizable:* For all $x \in X$, $A(x)$ contains an open set that contains the origin $\mathbf{0}$. This implies that for any $x_I$, $x_G$, there exists a finite-time acceleration control that solves the kinodynamic planning problem if $X_{free} = X$ (no obstacles).
2) *Rest-to-rest:* These problems require that for $x_I = (q_I, \dot{q}_I)$ and $x_G = (q_G, \dot{q}_G)$, $\dot{q}_I = \dot{q}_G = \mathbf{0}$.
3) *nD-double-integrator:* Each $q_i$ is independently actuated within global acceleration bounds $a_{min,i} < 0$ and $a_{max,i} > 0$. In this case, $A(x)$ is an axis-aligned rectangle that contains the origin, fixed for all $x \in X$.

To address optimality, a *cost functional* is defined as

$$L(\tilde{a}) = \int_0^{t_F} l(x(t), \tilde{a}(t))dt + l_F(x(t_F)). \tag{1}$$

We mainly consider time optimality, in which case $l(x, a) = 1$, and $l_F = 0$ if $x(t_F) = x_G$; otherwise, $l_F(t_F) = \infty$.

## III. TIME-OPTIMAL ACCELERATIONS

The task in this section is to calculate time-optimal solutions to the kinodynamic problem for the $n$D double integrator model and no obstacles: $X_{free} = X = \mathbb{R}^{2n}$. The solutions will work out so that controls are piecewise constant,

$$\tilde{a} = ((a_1, t_1), (a_2, t_2), \ldots, (a_n, t_n)), \tag{2}$$

which means that each $a_i$ is applied for duration $t_i$, starting at time $t_1 + t_2 \ldots t_{i-1}$. Initially, $a_1$ is applied at time $t = 0$.

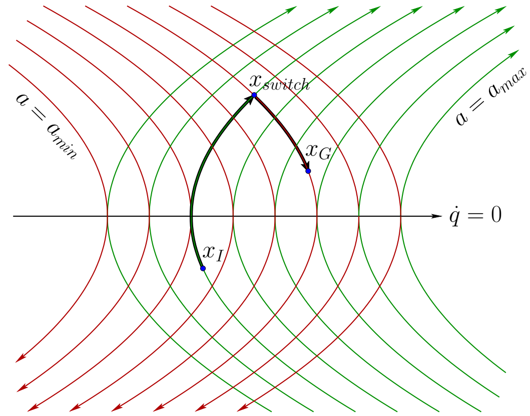



Fig. 1. Intersecting parabolas and traveling forward in time produces the time-optimal trajectory in the phase plane.

### A. Time-optimal control of a double integrator

Consider one double integrator, for which every $q \in \mathbb{R}$. The allowable accelerations form a closed interval $A = [a_{min}, a_{max}]$, in which $a_{min} < 0$ and $a_{max} > 0$. It is a control system of the form $\ddot{q} = a$ for $a \in [a_{min}, a_{max}]$ and has an associated *phase plane*, with coordinates $(q, \dot{q}) \in \mathbb{R}^2$. The task is to determine an acceleration control $\tilde{a} : [0, t_F] \to A$ for which $\Phi(x_I, \tilde{a}) = x_G$ and $t_F$ is as small as possible.

Pontryagin's maximum principle provides necessary conditions on the time-optimal trajectory by considering a co-state vector $(\lambda_1, \lambda_2)$ that serves as a generalized Lagrange multipliers for the constrained optimization problem [25]. Following the standard theory, the Hamiltonian is defined as

$$H(x, a, \lambda) = 1 + \lambda_1 x_2 + \lambda_2 a, \tag{3}$$

in which $x_1 = q$ and $x_2 = \dot{q}$, and the optimal $\tilde{a}$ is constrained to

$$\tilde{a}^*(t) = \operatorname*{argmin}_{a \in A} \{1 + \lambda_1(t)x_2(t) + \lambda_2(t)\tilde{a}(t)\}. \tag{4}$$

Solving the adjoint equation $\dot{\lambda}_i = -\frac{\partial H}{\partial x_i}$ results in $\lambda_1(t) = c_1$ and $\lambda_2(t) = c_2 - c_1 t$ for unknown constants $c_1$ and $c_2$. If $\lambda_2(t) < 0$, then $\tilde{a}^*(t) = a_{max}$, and if $\lambda_2(t) > 0$, then $\tilde{a}^*(t) = a_{min}$. Thus, the action may be assigned as $\tilde{a}^*(t) = -\mathrm{sgn}(\lambda_2(t))$, if $\lambda_2(t) \neq 0$. At the boundary case in which $\lambda_2(t) = 0$, any $a \in A$ may be chosen. Since $\lambda_2(t)$ is linear, it may change signs at most once, implying that the optimal control involves at most two "bangs", each corresponding to an extremal acceleration applied over a bounded time interval. Thus, the time-optimal control is of the form $\tilde{a}^* = ((a_1, t_1), (a_2, t_2))$, with degenerate possibilities of $t_1 = 0$ or $t_2 = 0$.

Further algebraic analysis is needed to precisely determine $a_1$, $t_1$, $a_2$, and $t_2$ for a given $(q_I, \dot{q}_I)$ and $(q_G, \dot{q}_G)$. See Figure 1. If a constant acceleration $a$ is applied at any phase $(q_0, \dot{q}_0)$, then a parabolic curve is traced out in the phase plane. If $q_0 = \dot{q}_0 = 0$, then its equation is $q = \frac{1}{2}\dot{x}^2$. More generally, it satisfies $q - \frac{1}{2}\dot{q}^2 = c$, in which the coefficient $c = q_0 - \frac{1}{2}\dot{q}_0^2$ corresponds to the parabola's intersection with the $\dot{q} = 0$ axis.

Next consider four parabolas based on all combinations of initial and goal states, and extremal controls. Let $I^+$ denote the parabola obtained from setting $q_0 = x_I = (q_I, \dot{q}_I)$ and applying constant control $a_{max}$; let $c(I^+)$ denote its $\dot{q} = 0$ intercept. Similarly, $I^-$ is obtained by applying $a_{min}$. Furthermore, $G^+$ and $G^-$ are obtained by setting $q_0 = x_G$ and applying $a_{max}$ and $a_{min}$, respectively. Assuming $x_I \neq x_G$, consider all possible intersections of $I^+$, $I^-$, $G^+$ and $G^-$. There are only two possible types: $I^+G^-$ and $I^-G^+$. The first results in $a_{max}$ applied until the intersection point, $x_{switch} = (q_{switch}, \dot{q}_{switch})$, is reached, followed by applying $a_{min}$. The second type applies $a_{min}$ first, followed by $a_{max}$.

The type $I^+G^-$ intersection occurs if $c(I^+) > c(G^-)$. The intersection position is the midpoint $q_{switch} = (c(I^+) + c(G^-))/2$, and the velocity is $\dot{q}_{switch} = \sqrt{2(q_{switch} - c(I^+))}$. Similarly, the type $I^-G^+$ intersection occurs if $c(I^-) < c(G^+)$. The intersection position is the midpoint $q_{switch} = (c(I^-) + c(G^+))/2$, and the velocity is $\dot{q}_{switch} = -\sqrt{2(q_{switch} - c(G^+))}$.

To determine the control timings to go from $x_I$ to $x_{switch}$ to $x_G$ note that changing velocity by an amount $d$ with constant acceleration $a$ requires time $d/|a|$. The timings are:

$$t_1 = (\dot{q}_{switch} - \dot{q}_I)/a_1 \tag{5}$$

and

$$t_2 = (\dot{q}_I - \dot{q}_{switch})/a_2, \tag{6}$$

respectively.

There will always be at least one intersection, but sometimes there are both $I^+G^-$ and $I^-G^+$. In this case, $t_1$ or $t_2$ may be negative for one intersection, but the other intersection provides a valid control. It is also possible to obtain valid controls for both cases, in which case the one that requires least time must be selected (Figure 2 will provide more details).

**Proposition 1** *The optimal control is $((a_1, t_1), (a_2, t_2))$, in which $a_1 = a_{min}$ and $a_2 = a_{max}$, or $a_1 = a_{min}$ and $a_2 = a_{max}$, and the durations $t_1$ and $t_2$ are given by (5) and (6).*

### B. Carefully waiting for multiple double integrators

To extend the result to a vector of double integrators, a time-wasting method is needed so that each double integrator arrives to its goal in its phase plane at the same time. This problem was considered in [8], [17], but only for the limited case in which their final velocities are zero. It was also considered in [13], but the gap problem, introduced shortly, was neglected. In this section, we introduce an explicit, complete, and computationally efficient solution to the general problem of time-optimal steering of $n$ independent integrators for any initial and goal state pairs in their respective phase planes in $O(n \lg n)$ time.

For a fixed $x_I$ and $x_G$, let $t^*$ be the time to reach the goal by applying the bang-bang solution of Section III-A. Now consider some $t_w > t^*$. Does a control $\tilde{u}$ necessarily exist that will cause $x_G$ to be reached at exactly time $t_w$?

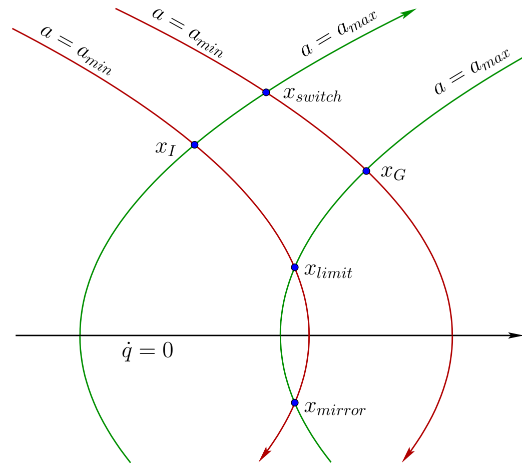



Fig. 2. If both $I^+G^-$ and $I^-G^+$ intersections occur, then there is a gap interval, over which waiting is not allowed.

The answer is yes if there is only one intersection type ($I^+G^-$ or $I^-G^+$). However, if both intersections occur, then the situation depicted in Figure 2 occurs (or its symmetric equivalent for $\dot{q} < 0$). The path from $x_I$ to $x_{limit} = (q_{limit}, \dot{q}_{limit})$ to $x_G$ corresponds to the slowest trajectory that reaches $x_G$ while remaining in the $\dot{q} > 0$ half-plane. The critical switching point $x_{limit}$ can be calculated using the parabola intersection algebra from Section III-A. The time taken by this trajectory is calculated as

$$t_{limit} = (\dot{q}_{limit} - \dot{q}_I)/a_{min} + (\dot{q}_G - \dot{q}_{limit})/a_{max}. \tag{7}$$

Thus, if $t_w \in [t^*, t_{limit}]$, then a solution exists (and requires only two constant control segments). If $t_w > t_{limit}$, then $x_G$ can no longer be reached while remaining in the $\dot{q} > 0$ half-plane. The next available time is obtained by continuing to apply $a = a_{min}$ until the second parabolic intersection point, called $x_{mirror}$ is reached, in the $\dot{q} < 0$ half-plane. Note that $x_{mirror} = (q_{limit}, -\dot{q}_{limit})$. Once $x_{mirror}$ is reached, $a = a_{max}$ is applied to arrive optimally at $x_G$. The time required to traverse this trajectory is

$$t_{mirror} = t_{limit} + 2\dot{q}_{limit}/a_{max} - 2\dot{q}_{limit}/a_{min}. \tag{8}$$

Thus, there may generally be a *gap interval* $(t_{limit}, t_{mirror})$ for which no solution exists.

Now suppose that $t_w \geq t^*$ and there is no gap interval, or $t_w \in [t^*, t_{limit}] \cup [t_{mirror}, \infty)$. A two-piece control $((a_1, t_1), (a_2, t_2))$ can be calculated as a solution to the following equations corresponding to the boundary conditions:

$$\dot{q}_I t_1 + \frac{a_1 t_1^2}{2} + (\dot{q}_I + a_1 t_1) t_2 + \frac{a_2 t_2^2}{2} = q_G - q_I, \tag{9}$$

$$a_1 t_1 + a_2 t_2 = \dot{q}_G - \dot{q}_I, \tag{10}$$

while satisfying $a_1, a_2 \in [a_{min}, a_{max}]$. Note that $t_2 = t_w - t_1$ and $0 \leq t_1, t_2 \leq t_w$. The solution is usually not unique, and can be selected either arbitrarily or by optimizing a relevant optimization objective, such as energy.

**Proposition 2** *The waiting method correctly generates a control that requires time $t_w$ to reach $x_G$ from $x_I$ or reports failure if a solution does not exist ($t_w$ lies in the gap interval).*

```
BANG_BANG_RRT_BIDIRECTIONAL(x_I, x_G)
1   T_a.init(x_I); T_b.init(x_G)
2   for i = 1 to K do
3       x_n ← BANG-BANG-NEAREST(S_a, α(i))
4       x_s ← BANG-BANG-STEER(x_n, α(i))
5       if x_s ≠ x_n then
6           T_a.add_vertex(x_s)
7           T_a.add_edge(x_n, x_s)
8           x'_n ← BANG-BANG-NEAREST(S_b, x_s)
9           x'_s ← BANG-BANG-STEER(x'_n, x_s)
10          if x'_s ≠ x'_n then
11              T_b.add_vertex(x'_s)
12              T_b.add_edge(x'_n, x'_s)
13          if x'_s = x_s then return SOLUTION
14      if |T_b| > |T_a| then SWAP(T_a, T_b)
15  return FAILURE
```

Fig. 3. Bidirectional RRT with bang-bang steering and quasimetric.

Now suppose that the optimal times have been calculated for $n$ double integrators to start at some $x_I$ and end at some $x_G$. In the worst case, every double integrator could have a gap interval. The problem is to find the smallest time $t$ such that $t \geq t^*$ and $t \notin (t_{limit}, t_{mirror})$. If a double integrator has no gap interval, then that part of the condition is dropped. A simple line sweeping algorithm [6] solves the problem as follows. Sort the optimal, limit, and mirror times for all double integrators into a single array. Sweep incrementally across the array, starting from the shortest time. In each step increment or decrement a counter of the number of double integrators that have a solution; $t_{limit}$ causes decrementing and the other two cases cause incrementing. The method terminates with the optimal $t$ when all $n$ integrators have a solution. The method takes time $O(n \lg n)$ time due to the initial sorting. Thus:

**Proposition 3** *The line-sweep algorithm solves the optimal waiting problem in $O(n \lg n)$ time for $n$ double integrators.*

## IV. PLANNING METHODS

### A. Kinodynamic RRT with bang-bang metric and steering

Figure 3 presents an outline of a balanced bidirectional RRT-based planning algorithm that uses bang-bang methods for both the metric and the steering method. A single-tree *goal-bias* algorithm could alternatively be made [22]. Let $\alpha(i) \in X$ denote the random state obtained in iteration $i$ ($\alpha$ could alternatively be a deterministic sequence that is dense in $X$ [21]). Line 3 returns the nearest state $x_n$ among all points $S_a$ visited by tree $T_a$. Using the tools from Section III, there are two natural choices for the (quasi)metric, $\rho(x, x')$, which is an estimate of the distance from $x$ to $x'$. Note it is not symmetric for our problem. The first choice $\rho_1(x, x')$ is the maximum time $t_1 + t_2$ from (5) and (6), taken over all $n$ double integrators. A slower and more accurate metric is $\rho_2(x, x')$ is the time with waiting, $t_w$, from Section III-B, which it takes for every double integrator to arrive at $x'$.

Line 4 is the time-optimal steering method from Section III-B. Collision checking is performed along the trajectory, and the steering stops at $x_s$ if an obstacle is hit or $\alpha(i)$ is reached. The new trajectory is added to $T_a$ (and $S_a$). In practice, this was accomplished in our experiments by inserting into $T_a$ nodes and edges along the trajectory; an exact method could alternatively be developed for representing and computing nearest points in $S_a$.

Lines 8 and 9 are similar to Lines 3 and 4, except that an attempt is made to connect the newest visited point $x_s$ to the nearest point $S_b$ in the other tree, $T_b$. Line 14 swaps the roles of the tree so that the smaller one explores toward $\alpha(i)$ and the larger one attempts to connect to the newly reached point.

### B. Bang-bang trajectory optimization

Let $X$, $X_{free}$, $x_I$, $x_G$, and $A$ be fixed for an $n$D double integrator system. Suppose that a piecewise-constant control $\tilde{a} : [0, t_F] \rightarrow A$ is given so that the resulting $\tilde{x} = \Phi(x_I, \tilde{a})$ is solution to Problem 2 from Section II. The task is to replace $\tilde{a}$ with a new control $\tilde{a}' : [0, t'_F] \rightarrow A$ so that $t'_F < t_F$ while maintaining the constraints that the trajectory maps into $X_{free}$ and arrives at $x_G$ at time $t'_F$. The new control $\tilde{a}'$ is constructed by selecting $t_1$ and $t_2$ such that $0 \leq t_1 < t_2 \leq t_F$.

For a given control, $\tilde{a}$, let $\tilde{a}[t_1, t_2]$ denote its restriction to the interval $[t_1, t_2]$. Thus, $\tilde{a}[t_1, t_2] : [t_1, t_2] \rightarrow A$.[1] The original control $\tilde{a}$ can be expressed as a sequence of three controls $\tilde{a}[0, t_1]$, $\tilde{a}[t_1, t_2]$, and $\tilde{a}[t_2, t_F]$ by applying $\Phi$ to each in succession. We replace the middle portion with a bang-bang control $\tilde{a}'[t_1, t'_2]$ using the methods of Section III-B. Since the method is time-optimal, it is known that $t'_2 \leq t_2$ (they are equal only if $\tilde{a}[t_2, t_F]$ is already time-optimal). The new control must satisfy

$$\Phi(\Phi(x_I, \tilde{a}[0, t_1])(t_1), \tilde{a}[t_1, t_2])(t_2) = \Phi(\Phi(x_I, \tilde{a}[0, t_1])(t_1), \tilde{a}'[t_1, t'_2])(t'_2), \quad (11)$$

which implies that the new control sequence $\tilde{a}[0, t_1]$, $\tilde{a}[t_1, t'_2]$, and $\tilde{a}[t_2, t_F]$ arrives at $x_G$ at time $t_f - (t_2 - t'_2)$. Fresh collision checking is needed for the trajectory from $\Phi(x_I, \tilde{a}[0, t_1])(t_1)$ to $\Phi(\Phi(x_I, \tilde{a}[0, t_1]), \tilde{a}'[t_1, t'_2])(t'_2)$.

The general template for *iterative bang-bang optimization* is:

1) Choose $t_1$ and $t_2$ according to a random or deterministic rule.
2) Attempt to replace $\tilde{a}[t_1, t_2]$ with the bang-bang alternative $\tilde{a}'[t_1, t'_2]$. If the result is collision free, then update $\tilde{a}$ with the modified control.
3) Go to Step 1, unless a termination criterion is met based on the number of iterations without any significant time reduction.

The rule of choosing $t_1$ and $t_2$ should produce a dense sequence of intervals in the following sense: the points of the form $(t_1, t_2) \in \mathbb{R}^2$ must be dense in the triangular region satisfying $0 \leq t_1 \leq t_2 \leq t_F$ (this ignores that fact that $t_F$ decreases in each iteration, and such out-of-bounds intervals

[1] The restrictions will be closed intervals that allow single-point overlaps, but this will not affect the resulting trajectories.

can be rejected in the analysis). A simple but effective rule is to first pick $t_1$ and $t_2$ uniformly at random. If $t_1 < t_2$, then replace $\tilde{a}[t_1, t_2]$. Otherwise, toss an unbiased coin to replace either $\tilde{a}[0, t_2]$ or $\tilde{a}[t_1, t_F]$. This extra consideration over purely random pairs (e.g., as in [13]) helps focus on the ends. Alternatively, $t_1$ and $t_2$ could be picked according to deterministic sequences such as Halton [12] to ensure convergence.

The termination criterion could be based on a hard limit on the number of iterations, or failure statistics (for example, no significant improvement more than $\epsilon > 0$ has occurred in the past 50 iterations).

**Proposition 4** *Using a dense interval sequence, iterative bang-bang optimization is asymptotically converges to a locally time-optimal trajectory.*

Note that the approach is not a variational optimization as in a gradient descent in path space [3]; it more resembles path shortening for basic path planning (called *shortcutting* in [10], [26]). This notion of optimality means that for any $\epsilon$, there exists a number of iterations such that from any $t_1$ and $t_2$, the computed control $\tilde{a}[t_1, t_2]$ cannot be further improved by time $\epsilon$. This is, however, weaker than is produced by shortcutting for basic path planning because the velocities at which obstacles are grazed depend on the sampling sequence, resulting in varying sequence-dependent solutions; this will be explained in Section V.

### C. Basic path planning with bang-bang state-space lifting

Consider taking the output of a basic path planning, lifting it into the state space using bang-bang control, and then applying the bang-bang optimization method of Section IV-B. Suppose we are given a kinodynamic planning problem for $n$D double integrators for which $x_I$ and $x_G$ are both at rest (zero velocity). Thus, $x_I = (q_I, \mathbf{0})$ and $x_G = (q_G, \mathbf{0})$. The first step is to compute a piecewise linear path $\tau : [0, 1] \to \mathcal{C}_{free}$, in which $\mathcal{C}_{free}$ is the projection of $X_{free}$ onto the configuration space. This could, for example, be computed by RRT-Connect [20], but the particular planner is unimportant.

We then introduce the *bang-bang transform*, described here for $\mathbb{R}^n$ and $a_{max} = -a_{min} = 1$ (it easily generalizes; see also [13]). For each vertex $q$ along the path $\tau$, extend it to $x = (q, \mathbf{0}) \in X_{free}$. For each edge between consecutive vertices, $q$, $q'$, execute a bang-bang control; we require that it is constrained to the edge and steers from $(q, \mathbf{0})$ to $(q', \mathbf{0})$. Let $v = q' - q$, normalized as $\hat{v} = v/\|v\|$. Let $s = \max_i(|\hat{v}_i|)$. Let $a_i = \hat{v}_i/s$ and $t = \sqrt{s\|v\|}$. The bang-bang control is $((a, t), (-a, t))$. This transform is applied to each edge of the path and the resulting controls are concatenated.

The following were straightfoward to show:

**Proposition 5** *The bang-bang transform of a path is time-optimal. Furthermore, $\Phi(x_I, \tilde{a}) = x_G$ and the resulting trajectory is collision free.*

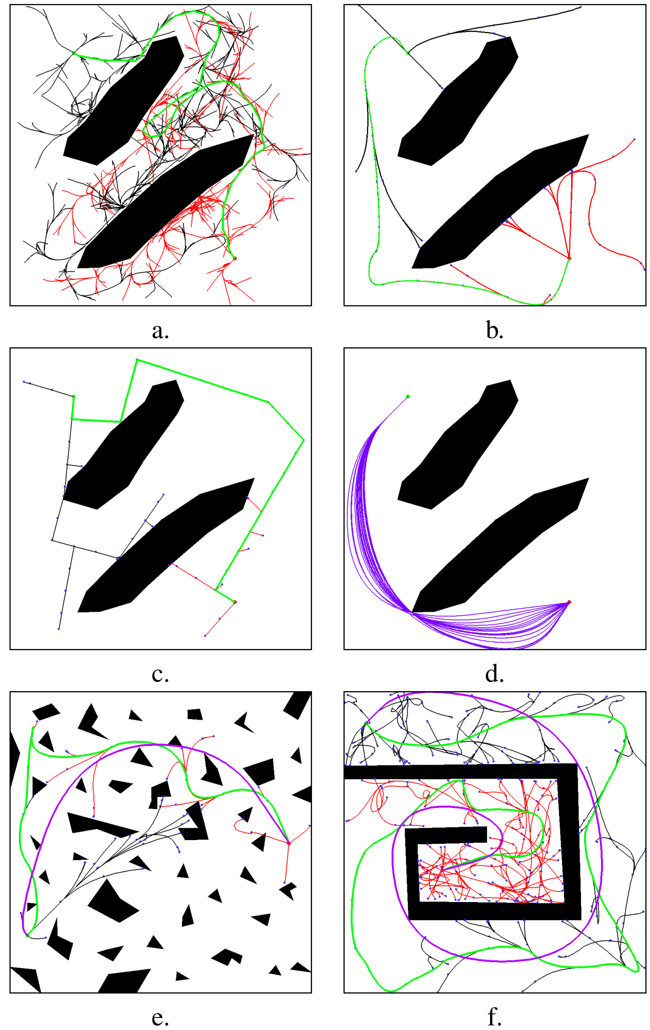



Fig. 4. a) Original kinodyamic RRT-Bi [22], b) The proposed BB-RRT, c) RRT-Connect [20] applied to the 2D projection, d) multiple bang-bang optimizations of a planned path, e & f) two more BB-RRT examples with BB-optimized paths (purple).

**Proposition 6** *using the bang-bang transform, any piecewise-linear solution to Problem 1 can be converted via the bang-bang transform into a corresponding solution to Problem 2, restricted to double integrators and rest-to-rest.*

Note that if a solution exists to a basic path planning problem, then a piecewise linear solution must also exist.

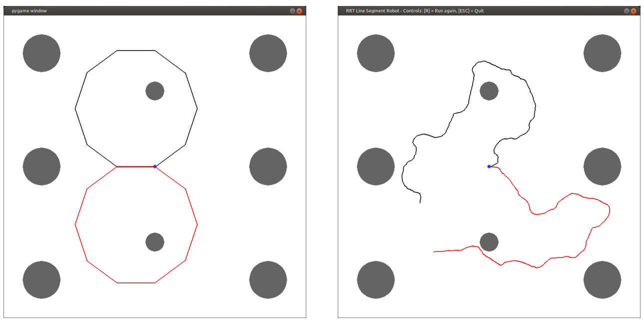
## V. EXPERIMENTS

The algorithms were implemented in Python 3.9.5 on a Windows 10 PC with an AMD Ryzen 7 5800X CPU and 32GB 3200Mhz CL16 RAM. Naive methods were used for nearest neighbor searching and collision detection. Most examples use a four-dimensional state space corresponding to a 2D workspace in which a planar vehicle moves with double intregator dynamics. The last examples involve a planar manipulator with up to 1000 degrees of freedom, resulting in a 2000-dimensional state space. All examples assign $a_{max} = -a_{min} = 1$.

Figure 4 compares the new BB-RRT method (Figure 3) to the original kinodynamic RRT [22] and RRT-Connect [20]

| Method | RunTime | Nodes | ColCh | $\tilde{x}$-Time |
|---|---|---|---|---|
| RRT-Bi | 27.013 | 1589.1 | 4364.8 | 311.53 |
| BB-RRT | 0.017276 | 57.772 | 599.48 | 126.99 |
| RRT-Con | 0.004738 | 60.663 | 713.25 | - |
| BB-Opt | 0.021547 | - | 3291.3 | 72.452 |

Fig. 5. Statistics averaged over 1000 runs.



a. b.

Fig. 6. BB-optimization applied to a planar manipulator with dynamics.

on the 2D projection that ignores velocities and dynamics. The state space $X$ is $[0, 800]^2 \times [-10, 10]^2$. Statistics are reported in Figure 5. The $\rho_1$ metric was used. The $S_a$ and $S_b$ sets were approximated by placing new RRT nodes along long edges for every 12 collision checks (see Section 5.5.2 of [21]). The BB-RRT is about 1564 times faster on average than the original RRT. RRT-Connect is even faster, but it only constructs paths on the 2D configuration space, and the number of collision checks is comparable. Figure 4.d shows 50 results for the bang-bang optimizer of Section IV-B, applied to the same initial path; computation times are fairly consistent across runs and problems, depending mainly on path length and collision detector cost.

The BB-RRT has the advantage, much like RRT-Connect, in that there are no parameters to tune. The original RRT has parameters for the stepping size, the discrete set of actions, and the connection distance (the two trees do not exactly meet). For the example in Figure 4.a, we used 24 constant acceleration actions, $\Delta t = 5$, and connection distances of $\Delta q = 5$ and $\Delta \dot{q} = 2$. In the metric, the velocity components were weighted 17.32 times more than the configuration components. Figures 4.e and 4.f show two more examples, for which BB-RRT took on average 0.0368s and 0.4072s, respectively, over 1000 runs. The speedup factors over RRT-Bi were 837.6 and 368.5. Once again, RRT-Connect on the 2D projection was faster, by factors 6.65 and 12.5, respectively. Bang-bang optimized paths are shown in purple, the original paths in green.

Figure 6 depicts additional experiments, performed for an $n$-link, fixed-base planar manipulator, modeled as a kinematic chain of line segments of equal length. The initial and goal configurations form a regular polygon, as shown in Figure 6.a for $n = 10$ links. Paths were initially computed using RRT-Connect on $\mathcal{C}$ and then lifted into $X$ using the bang-bang transform of Section IV-C. Each joint has limits $\pm\pi$ and is modeled as a double integrator. The BB-Optimization method was applied computed paths from $n = 10$ (dimension of $X$ is 20) up to $n = 1000$ (dimension of $X$ is 2000). Figure 6.b shows configurations in the RRTs that were grown from initial and goal configurations, respectively. Average running times (10 runs) to fully converge are 1.63s for 10 links, 1.72s for 20 links, 3.75s for 50 links, 20.75s for 100 links, and 1123.35s for 1000 links (rapid increases due to dimension were caused by a naive quadratic-time implementation of the waiting method, rather than the $O(n \lg n)$ method presented in Section III-B). Termination was reached if 200 iterations were attempted with no more than 0.1s reduction in trajectory time; the most dramatic reductions occur in the first early iterations. In a typical run for 100 links, the trajectory execution time was reduced from 57.95s to 8.46s.

## VI. DISCUSSION

We have proposed, analyzed, and implemented methods that accelerate planning performance and optimize solutions. The key is to quickly compute bang-bang time-optimal controls using analytic solutions, to produce both metrics and steering methods. Although the study has been limited to RRTs, we expect it could enhance other sampling-based planning methods that rely on distance metrics or steering, such as *probabilistic roadmaps* [1], [19] or *expansive space trees* [16]. One of the key observations of our experiments is that plan-and-optimize is superior when applicable: It is more reliable to explore the C-space first, lift the solution into the state space, and then use bang-bang optimization. However, this applies only for rest-to-rest problems; for more general problems, a bang-bang enhanced RRT could be applied to bring each of $x_I$ and $x_G$ to zero velocity by biasing samples to the $(q, \mathbf{0})$ plane.

The encouraging results of this paper lead naturally to many new questions and further studies. The computed examples presented here were limited to double integrators, implying that $A(x)$ is a constant rectangle for all $x \in X$. We are currently adapting the steering methods for acceleration bounds that vary with state, allowing for more general, fully actuated systems of the form $\ddot{q} = f(q, \dot{q}, u)$. Another important direction is to develop bang-bang boosted versions of asymptotically optimal planners, such as RRT* [18] and SST* [24]; this would enable stronger comparisons to the plan-and-optimize approach, both in computation time and solution quality. Also, improvements can be made to the iterative bang-bang optimization through strategic interval selection. Finally, efficient nearest-neighbor algorithms should be developed for the bang-bang metric over a tree of parabolic arcs (analogous to [2], [32]).

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